Vector Calculus:
Del Operator & Field Operations

EE3321
Electromagnetic Field Theory

Outline

• Scalar vs. Vector Fields
• The Del Operator
• Gradient of a Scalar Field
• Divergence of a Vector Field
• Curl of a Vector Field
• Laplacian Operation
Scalar Field Vs. Vector Field

Scalar Field, $f(x, y)$
- magnitude $(x, y, z)$

Vector Field, $\mathbf{v}(x, y)$
- magnitude $(x, y, z)$
- direction $(x, y, z)$

Isocontour Lines

Isocontour lines trace the paths of equal value. Closely space isocontours conveys that the function is varying rapidly.
The Del Operator

The del operator $\nabla$ is the vector differential operator. It is sort of a 3D derivative. Even though $\nabla$ is a vector, it is almost never written as $\vec{\nabla}$.

### Coordinate System

<table>
<thead>
<tr>
<th>Coordinate System</th>
<th>Del Operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian</td>
<td>$\nabla = \frac{\partial}{\partial x} \hat{a}_x + \frac{\partial}{\partial y} \hat{a}_y + \frac{\partial}{\partial z} \hat{a}_z$</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>$\nabla = \frac{\partial}{\partial \rho} \hat{a}<em>\rho + \frac{1}{\rho} \frac{\partial}{\partial \theta} \hat{a}</em>\theta + \frac{\partial}{\partial z} \hat{a}_z$</td>
</tr>
<tr>
<td>Spherical</td>
<td>$\nabla = \frac{\partial}{\partial r} \hat{a}<em>r + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{a}</em>\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{a}_\phi$</td>
</tr>
</tbody>
</table>

Derived from the Cartesian equation using coordinate transformation.

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### Summary of Vector Operations

<table>
<thead>
<tr>
<th>Operation</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector Addition &amp; Subtraction $\vec{U} \pm \vec{V}$</td>
<td>Vectors</td>
<td>Vector</td>
</tr>
<tr>
<td>Dot Product $\vec{U} \cdot \vec{P}$</td>
<td>Vectors</td>
<td>Scalar</td>
</tr>
<tr>
<td>Cross Product $\vec{U} \times \vec{P}$</td>
<td>Vectors</td>
<td>Vector</td>
</tr>
<tr>
<td>Gradient $\nabla \vec{f}$</td>
<td>Scalar Function</td>
<td>Vector Function</td>
</tr>
<tr>
<td>Divergence $\nabla \cdot \vec{U}$</td>
<td>Vector Function</td>
<td>Scalar Function</td>
</tr>
<tr>
<td>Curl $\nabla \times \vec{U}$</td>
<td>Vector Function</td>
<td>Vector Function</td>
</tr>
<tr>
<td>Scalar Laplacian $\nabla^2 \vec{f}$</td>
<td>Scalar Function</td>
<td>Scalar Function</td>
</tr>
<tr>
<td>Vector Laplacian $\nabla^2 \vec{U}$</td>
<td>Vector Function</td>
<td>Vector Function</td>
</tr>
</tbody>
</table>
Gradient of a Scalar Field

We start with a scalar field…

\[ f(x, y) \]
Gradient of a Scalar Field (2 of 3)

…then plot the gradient on top of it. Color in background is the original scalar field.

$$\nabla f(x, y)$$

Gradient of a Scalar Field (3 of 3)

The gradient will always be perpendicular to the isocontour lines.
The Gradient

The gradient calculates how rapidly, and in what direction, a scalar function is increasing.

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<thead>
<tr>
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<tr>
<td>Cartesian</td>
<td>( \nabla V = \frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z )</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>( \nabla V = \frac{\partial V}{\partial \rho} \hat{a}<em>\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}</em>\phi + \frac{\partial V}{\partial z} \hat{a}_z )</td>
</tr>
<tr>
<td>Spherical</td>
<td>( \nabla V = \frac{\partial V}{\partial r} \hat{a}<em>r + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{a}</em>\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{a}_\phi )</td>
</tr>
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Algebra Rules for the Gradient

\[
\nabla (U \pm V) = \nabla U \pm \nabla V \quad \text{Sum/Difference Rule}
\]

\[
\nabla (UV) = U \nabla V + V \nabla U \quad \text{Product Rule}
\]

\[
\nabla \left( \frac{U}{V} \right) = \frac{V \nabla U - U \nabla V}{V^2} \quad \text{Quotient Rule}
\]

\[
\nabla V^n = nV^{n-1} \nabla V \quad \text{Power Rule}
\]
Properties of the Gradient

1. The gradient of a scalar function is a vector function.
2. The magnitude of $\nabla V$ is the local maximum rate of change in $V$.
3. $\nabla V$ points in the direction of maximum rate of change in $V$.
4. $\nabla V$ at any point is perpendicular to the constant $V$ surface that passes through that point.
5. $\nabla V$ points toward increasing numbers in $V$.

Divergence of a Vector Field
Divergence of a Vector Field (1 of 2)

Suppose we start with the following vector field...

\[ \mathbf{A}(x, y) \]

Divergence of a Vector Field (2 of 2)

We then plot the divergence as the color in the background. The arrows are the original vector function.

\[ \nabla \cdot \mathbf{A}(x, y) \]
Divergence

The divergence of a vector field is a scalar field that measures the tendency of a vector field to diverge from a point or converge to a point.

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<tr>
<td>Cartesian</td>
<td>( \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} )</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>( \nabla \cdot \vec{A} = \frac{1}{\rho} \frac{\partial (\rho A_{\rho})}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} )</td>
</tr>
<tr>
<td>Spherical</td>
<td>( \nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} )</td>
</tr>
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</table>

Algebra Rules for Divergence

\[
\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B} \quad \text{Distributive Rule}
\]
Properties of Divergence

1. The divergence of a vector function is a scalar function.
2. The divergence of a scalar field does not make sense.
3. The original vector field will point form larger to smaller numbers in the scalar field.

Curl of a Vector Field
Suppose we start with the following vector field…

\[ \vec{B}(x, y) \]

The color in the background is the magnitude of the curl. The direction is either into, or out of, the screen. Red indicates \( +z \) direction while blue indicates \( -z \) direction.
Curl

The curl of a vector function is a vector function that quantifies the tendency of the vector field to circulate around an axis. The magnitude of the curl conveys the strength of the circulation. The direction of the curl is the axis of the circulation.

<table>
<thead>
<tr>
<th>Coordinate System</th>
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<tr>
<td>Cartesian</td>
<td>( \nabla \times \vec{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{a}_x + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{a}_y + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{a}_z )</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>( \nabla \times \vec{A} = \left( \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{a}<em>\rho + \left( \frac{\partial A</em>\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{a}<em>\phi + \left( \frac{\partial A</em>\phi}{\partial \rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right) \hat{a}_\theta )</td>
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<tr>
<td>Spherical</td>
<td>( \nabla \times \vec{A} = \frac{1}{r \sin \theta} \left[ \frac{\partial (A_z \sin \theta)}{\partial \theta} \hat{a}<em>\phi - \frac{\partial A</em>\phi}{\partial \theta} \hat{a}<em>r + \frac{1}{r \sin \theta} \frac{\partial (r A</em>\phi)}{\partial \phi} \hat{a}<em>\theta + \frac{1}{r} \frac{\partial (\rho A</em>\rho)}{\partial \phi} \hat{a}_\phi \right] )</td>
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Algebra Rules for Curl

\( \nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B} \)  \hspace{1cm} \text{Distributive Rule} \\
\( \nabla \times (\vec{A} \times \vec{B}) = \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} \)
Properties of Curl

1. The curl of a vector function is a vector function.
2. The curl of a scalar field does not make sense.
3. Curl follows the right-hand rule.
4. The divergence of the curl of a vector field is always zero. \( \nabla \cdot (\nabla \times \vec{A}) = 0 \)
5. The curl of the gradient of a scalar field is always zero. \( \nabla \times (\nabla V) = 0 \)
6. Note \( \nabla \cdot \vec{A} \neq \vec{A} \cdot \nabla \). \( \nabla \cdot \vec{A} \) calculates the derivative of \( \vec{A} \), whereas \( \vec{A} \cdot \nabla \) sets up a derivative operation that is scaled by \( \vec{A} \).

Laplacian Operation
Scalar Laplacian

The scalar Laplacian is defined as the divergence of the gradient. It is sort of a measure of the tendency of the scalar function to form bowls.

\[ \nabla^2 V = \nabla \cdot (\nabla V) \]

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<td>[ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} ]</td>
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<td>[ \nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \theta^2} ]</td>
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<td>Spherical</td>
<td>[ \nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} ]</td>
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Visualization of Scalar Laplacian
Vector Laplacian

The vector Laplacian is defined as the gradient of the divergence minus the curl.

\[ \nabla^2 \vec{A} = \nabla \left( \nabla \cdot \vec{A} \right) - \nabla \times \nabla \times \vec{A} \]

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Visualization of Vector Laplacian

- \( \vec{A}(x,y) \)
- \( \nabla^2 \vec{A}(x,y) \)
Properties of the Laplacian

1. The Laplacian of a scalar function is a scalar function.
2. The scalar Laplacian arises when dealing with electric potential.
3. The Laplacian of a vector function is a vector function.
4. The vector Laplacian arises in the wave equation.

Classification of Vector Fields
**Irrotational**

A vector field is irrotational if \( \nabla \times \vec{A} = 0 \)

![Irrotational vs Rotational Diagram](image)

**Examples of Irrotational Fields**
- Gravity
- Electric potential

**Consequences:**
1. Irrotational fields must flow in essentially straight lines.
2. If \( \nabla \times \vec{A} = 0 \), then \( \oint_A \vec{A} \cdot d\vec{l} = 0 \) and the field is said to be **conservative**.

---

**Solenoidal (Divergenceless)**

A vector field is solenoidal if \( \nabla \cdot \vec{A} = 0 \)

![Solenoidal vs Divergence Diagram](image)

**Examples of Solenoidal Fields**
- Fluid flows
- Magnetic fields

**Consequences:**
1. If the vector function does not diverge from any points, it must form loops.
2. If \( \nabla \cdot \vec{A} = 0 \), then \( \iiint_S \vec{A} \cdot d\vec{s} = 0 \) and the field will have zero net flux.
A vector field is Laplacian if it is both irrotational and solenoidal

\[ \nabla \times \vec{A} = 0 \quad \nabla \cdot \vec{A} = 0 \]