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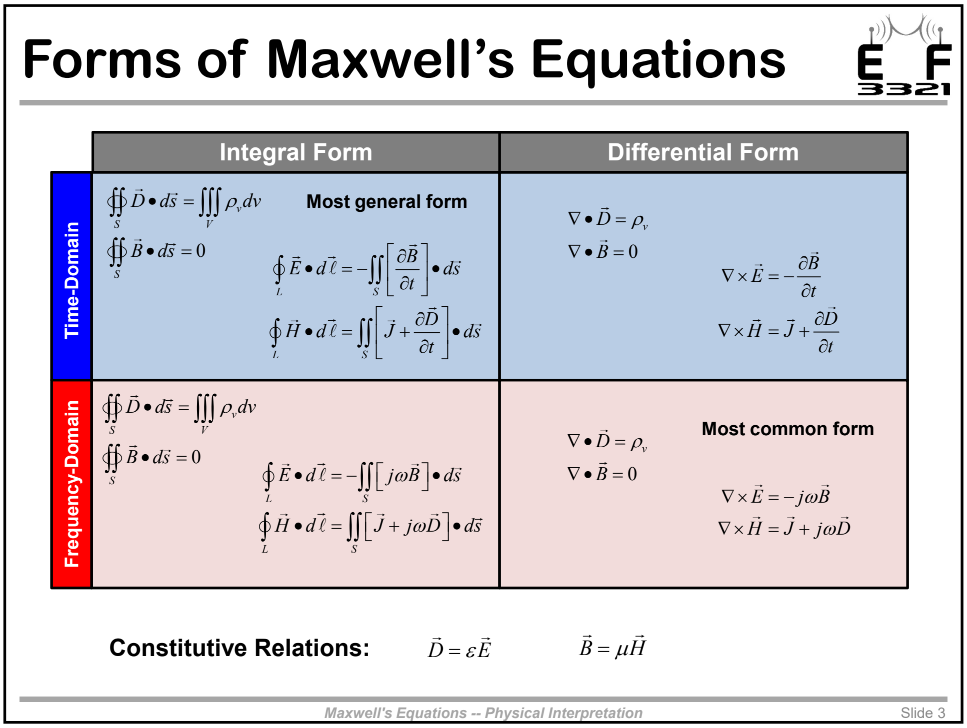
Maxwell's Equations:

Physical Interpretation

EE-3321
Electromagnetic Field Theory

Outline

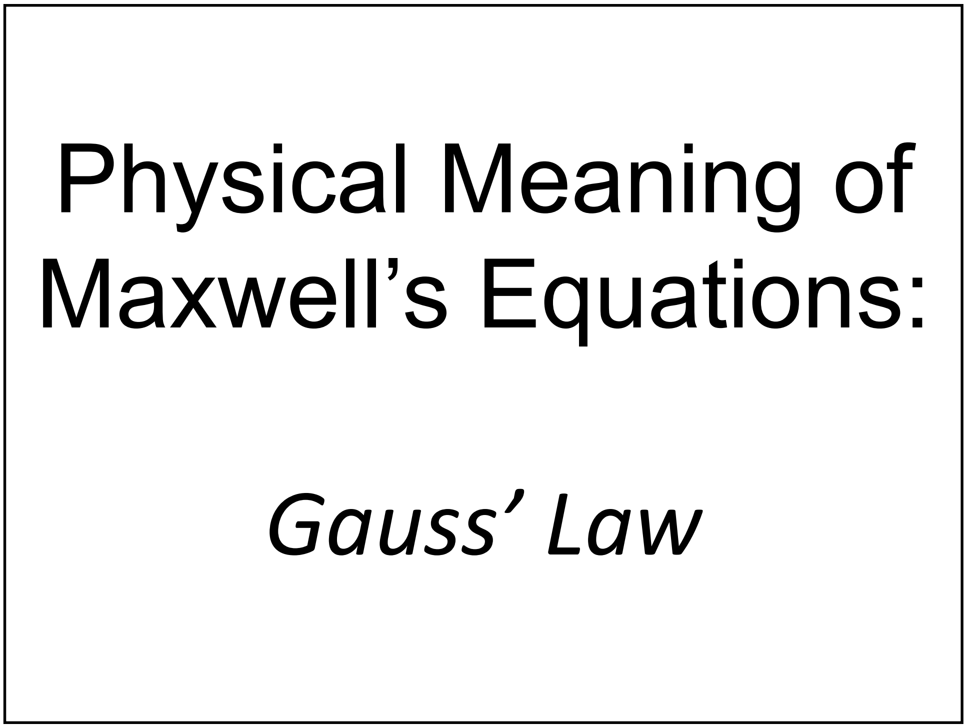
- Maxwell's Equations
- Physical Meaning of Maxwell's Equations
  - Gauss' law
  - Gauss' law for magnetic fields
  - Faraday's law
  - Ampere's circuit law
  - Constitutive relations



Forms of Maxwell's Equations

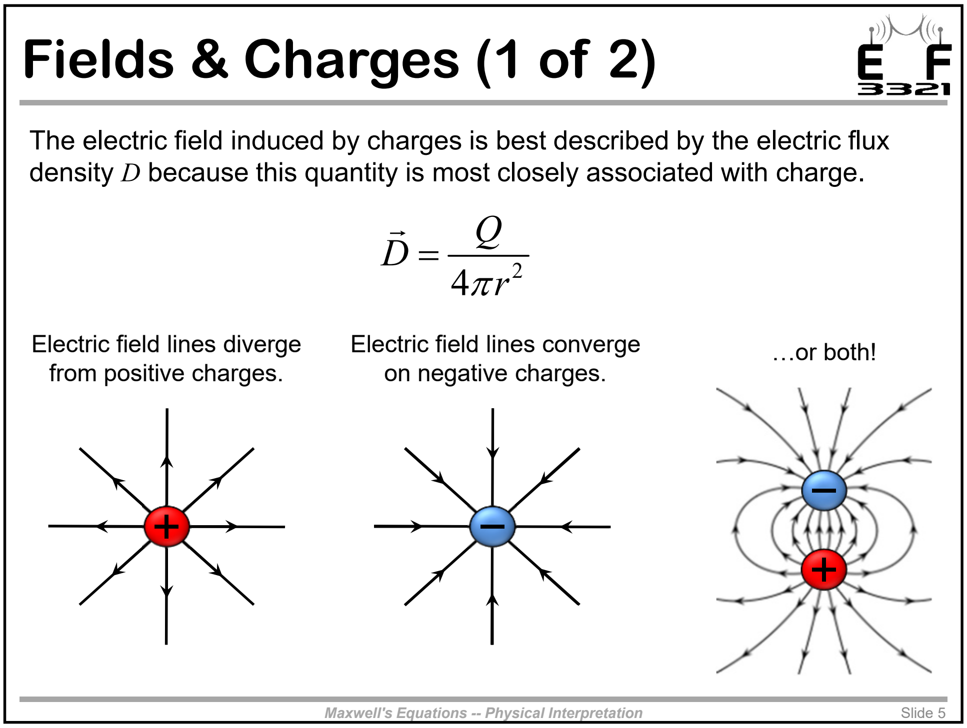
	Integral Form	Differential Form
Time-Domain	$\oiint_S \vec{D}\bullet d\vec{s} = \iiint_V \rho_v dv$ $\oiint_S \vec{B}\bullet d\vec{s} = 0$ $\oint_L \vec{E}\bullet d\vec{\ell} = -\iint_S \left[\frac{\partial \vec{B}}{\partial t}\right]\bullet d\vec{s}$ $\oint_L \vec{H}\bullet d\vec{\ell} = \iint_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right]\bullet d\vec{s}$ **Most general form**	$\nabla\bullet\vec{D} = \rho_v$ $\nabla\bullet\vec{B} = 0$ $\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\nabla\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
Frequency-Domain	$\oiint_S \vec{D}\bullet d\vec{s} = \iiint_V \rho_v dv$ $\oiint_S \vec{B}\bullet d\vec{s} = 0$ $\oint_L \vec{E}\bullet d\vec{\ell} = -\iint_S \left[j\omega\vec{B}\right]\bullet d\vec{s}$ $\oint_L \vec{H}\bullet d\vec{\ell} = \iint_S \left[\vec{J} + j\omega\vec{D}\right]\bullet d\vec{s}$	**Most common form** $\nabla\bullet\vec{D} = \rho_v$ $\nabla\bullet\vec{B} = 0$ $\nabla\times\vec{E} = -j\omega\vec{B}$ $\nabla\times\vec{H} = \vec{J} + j\omega\vec{D}$

Constitutive Relations: $\vec{D} = \varepsilon\vec{E}$ $\vec{B} = \mu\vec{H}$



Physical Meaning of Maxwell's Equations:

Gauss' Law



Fields & Charges (1 of 2)

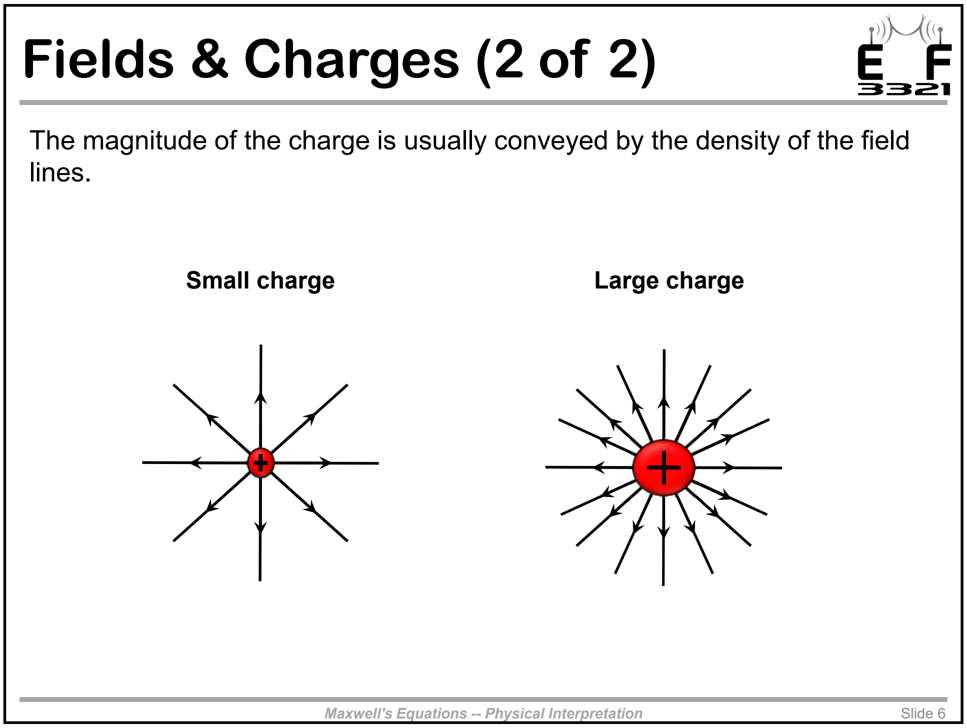
The electric field induced by charges is best described by the electric flux density D because this quantity is most closely associated with charge.

$$\vec{D} = \frac{Q}{4\pi r^2}$$

Electric field lines diverge from positive charges.

Electric field lines converge on negative charges.

…or both!



Fields & Charges (2 of 2)

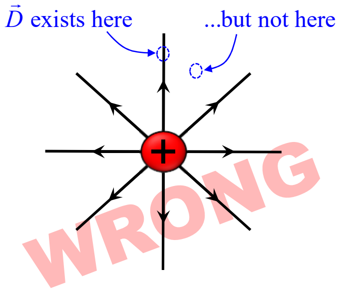
The magnitude of the charge is usually conveyed by the density of the field lines.

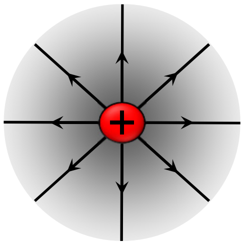
Small charge

Large charge

A Note About Field Lines

It is in some ways misleading to draw electric field lines because it incorrectly implies that the field exists at some points but not others.



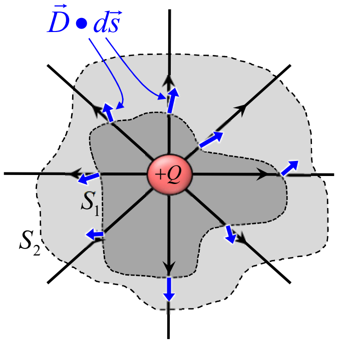


The electric field is a smooth and continuous phenomenon, but that is hard to convey in a diagram.

How to Calculate Total Charge Q

Method 1: Surface integral of electric flux

Since the density of field lines convey the amount of charge, it makes sense that we can calculate the total charge Q enclosed within a surface S by integrating the flux of the field lines at the surface.



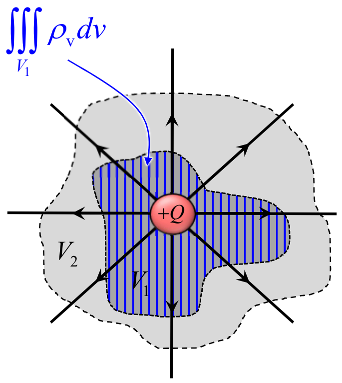
$$Q = \oiint_{S_1} \vec{D} \bullet d\vec{s} = \oiint_{S_2} \vec{D} \bullet d\vec{s}$$

The choice of surface S does not matter as long as the surface completely encloses the charge. Usually the shape of the surface is chosen to make the math as simple as possible.

How to Calculate Total Charge Q

Method 2: Volume integral of electric charge density

The total charge Q can be calculated by integrating the electric charge density ρ_v within a volume V that completely encompasses the charge.



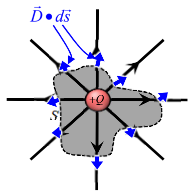
$$Q = \iiint_{V_1} \rho_v dv = \iiint_{V_2} \rho_v dv$$

The choice of volume V is the same as the choice of the surface for Method 1. the shape does not matter as along it completely encompasses the charge. It is usually chosen for mathematical convenience.

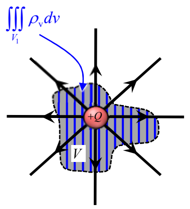
Gauss' Law in Integral Form

Both methods calculate the same total charge so they can be set equal.

Method 1



Method 2



$$\boxed{Q = \oiint_S \vec{D} \bullet d\vec{s} = \iiint_V \rho_v dv}$$

Apply Divergence Theorem

The *divergence theorem* allows us to write a closed-contour surface integral as a volume integral.

$$\oiint_S \vec{A} \bullet d\vec{s} = \iiint_V \left(\nabla \bullet \vec{A}\right) dv$$

Applying this to Gauss' law gives us

$$Q = \oiint_S \vec{D} \bullet d\vec{s} = \iiint_V \rho_{\mathrm{v}} dv$$

$$\downarrow$$

$$Q = \iiint_V \left(\nabla \bullet \vec{D}\right) dv = \iiint_V \rho_{\mathrm{v}} dv$$

Gauss' Law in Differential Form

If the surface S and volume V describe the same space, the argument of both integrals must be equal. Setting these arguments equal gives Gauss' law in differential form.

$$Q = \iiint_V \left(\nabla \bullet \vec{D}\right) dv = \iiint_V \rho_{\mathrm{v}} dv$$

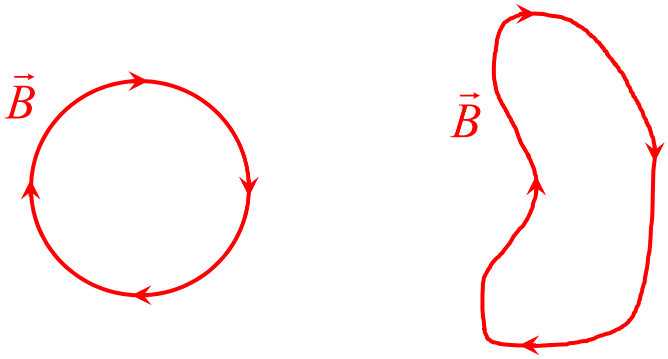
$$\boxed{\nabla \bullet \vec{D} = \rho_{\mathrm{v}}}$$

Physical Meaning of Maxwell's Equations:

Gauss' Law for Magnetic Fields

No Magnetic Charge

Since there exists no isolated magnetic charge, the magnetic field cannot have a beginning or an end. The magnetic field can only form loops.

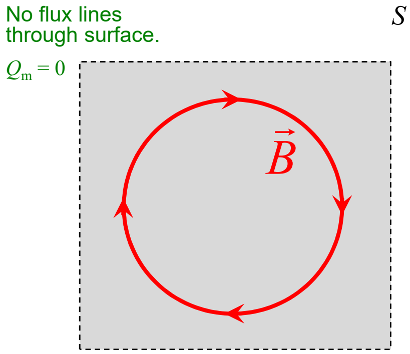


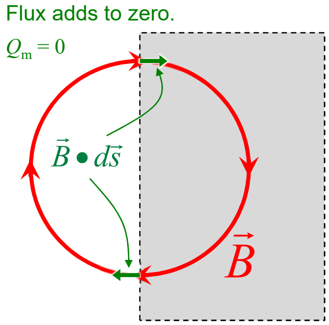
Prove Zero Magnetic Charge

Surface integral of magnetic flux

Following what we did for electric fields, we calculate total magnetic charge enclosed within a surface S by integrating the flux of the magnetic field lines at the surface.

$$0 = \oiint_S \vec{B} \bullet d\vec{s}$$





Gauss' Law for Magnetic Fields in Integral Form

The result from last slide is Gauss' law for magnetic fields.

$$\boxed{\oiint_S \vec{B} \bullet d\vec{s} = 0}$$

Apply Divergence Theorem

The *divergence theorem* allows us to write a closed-contour surface integral as a volume integral.

$$\oiint_S \vec{A} \bullet d\vec{s} = \iiint_V \left(\nabla \bullet \vec{A}\right) dv$$

Applying this to Gauss' law for magnetic fields gives us

$$Q_\text{m} = \oiint_S \vec{B} \bullet d\vec{s} = \iiint_V \rho_\text{m} dv$$

$$\downarrow$$

$$Q_\text{m} = \iiint_V \left(\nabla \bullet \vec{B}\right) dv = \iiint_V \rho_\text{m} dv$$

Gauss' Law for Magnetic Fields in Differential Form

If the surface S and volume V describe the same space, the argument of both integrals must be equal. Setting these arguments equal gives Gauss' law for magnetic fields in differential form.

$$Q_\text{m} = \iiint_V \left(\nabla \bullet \vec{B}\right) dv = \iiint_V \rho_\text{m} dv$$

$$\nabla \bullet \vec{B} = \rho_\text{m}$$

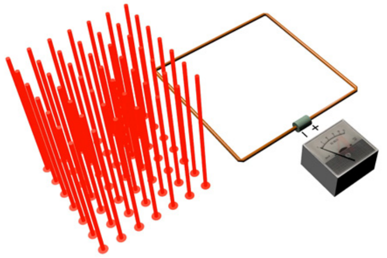
However, there is no magnetic charge so $\rho_\text{m} = 0$.

$$\boxed{\nabla \bullet \vec{B} = 0}$$

Physical Meaning of Maxwell's Equations:

Faraday's Law

Faraday's Experiment



Observations:

1. The more (or stronger) the magnetic flux, the higher the voltage reading.

$$V_{\text{emf}} \propto \psi \qquad \psi = \iint_S \vec{B} \bullet d\vec{s}$$

2. The more turns of the loop, the higher the voltage reading.

$$V_{\text{emf}} \propto N \qquad N \equiv \#\text{ turns}$$

3. The faster the time rate of change of the magnetic flux, the higher the voltage reading.

$$V_{\text{emf}} \propto \frac{d\psi}{dt}$$

Calculate Induced Voltage (1 of 2)

Method 1: By experiment.

Faraday performed an experiment and determined that

$$V_{\text{emf}} = -N\frac{d\psi}{dt}$$

The total magnetic flux ψ accounting for the number of turns N is

$$N\psi = \iint\limits_S \vec{B} \bullet d\vec{s}$$

$\psi \equiv \text{magnetic flux}$

$N\psi \equiv \text{magnetic flux linkage}$

Flux linkage is a property of a two-terminal device. It is not equivalent to flux. Flux can exist without the loop. Further, if the loops do not have the same orientation, they will not “link” to the magnetic field the same.

Flux and flux linkage are not the same thing, but often used synonymously because most devices are designed so that each loop links the same to the magnetic field and the math reduces to them being nearly equivalent.

Combing the above equations leads an expression for V_{EMF} in terms of just the magnetic flux density B.

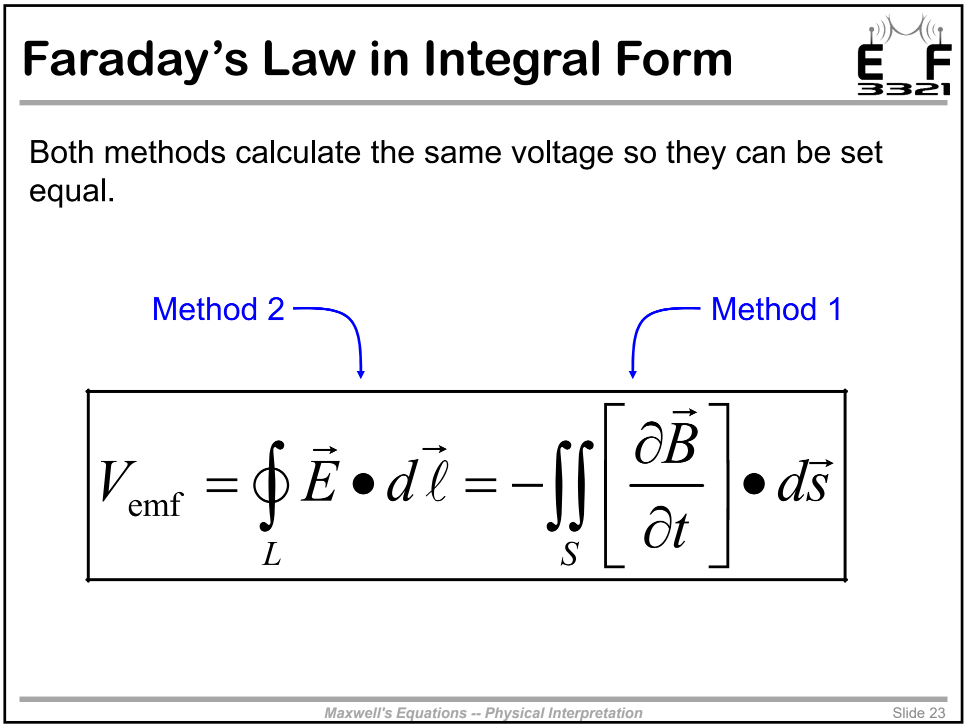
$$V_{\text{emf}} = -N\frac{d\psi}{dt} = -\frac{d}{dt}\left(N\psi\right) = -\frac{\partial}{\partial t}\iint\limits_S \vec{B} \bullet d\vec{s} = -\iint\limits_S \left[\frac{\partial \vec{B}}{\partial t}\right] \bullet d\vec{s}$$

Calculate Induced Voltage (2 of 2)

Method 2: Use Kirchoff’s voltage law

The voltage across the terminals of the resistor can be calculated using Kirchoff’s voltage law. For electromagnetics, Kirchoff’s voltage law becomes a line integral. Assuming the resistor is very small compared to the loop, we get

$$V_{\text{emf}} = \oint\limits_L \vec{E} \bullet d\vec{\ell} \qquad V = E \cdot \ell$$



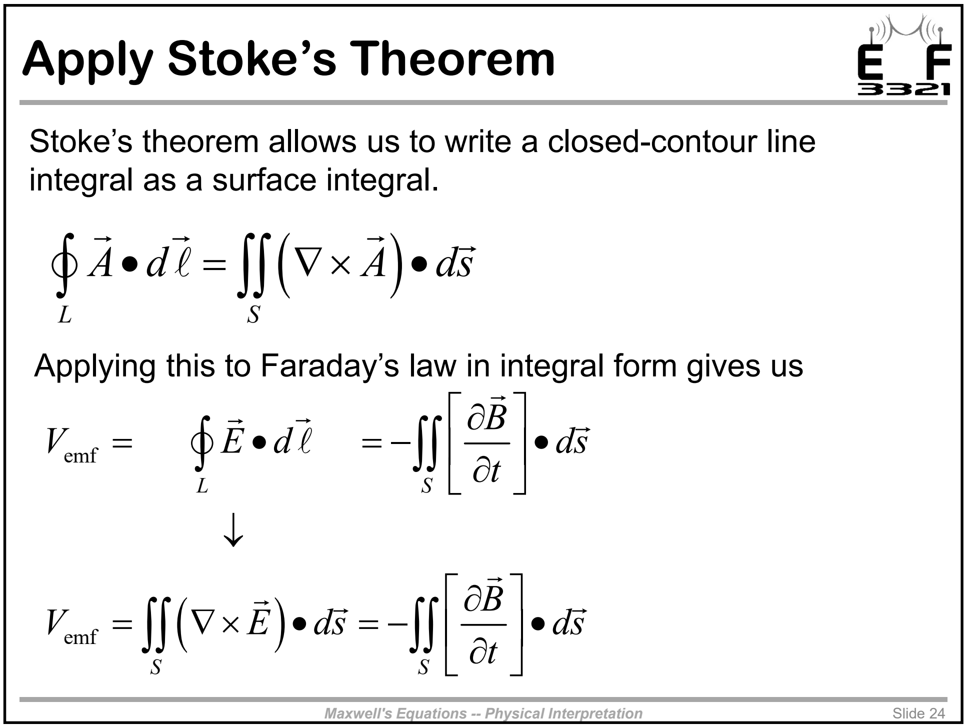
Faraday’s Law in Integral Form

Both methods calculate the same voltage so they can be set equal.

Method 2

Method 1

$$\boxed{V_{\text{emf}} = \oint_L \vec{E} \bullet d\vec{\ell} = -\iint_S \left[\frac{\partial \vec{B}}{\partial t} \right] \bullet d\vec{s}}$$



Apply Stoke’s Theorem

Stoke’s theorem allows us to write a closed-contour line integral as a surface integral.

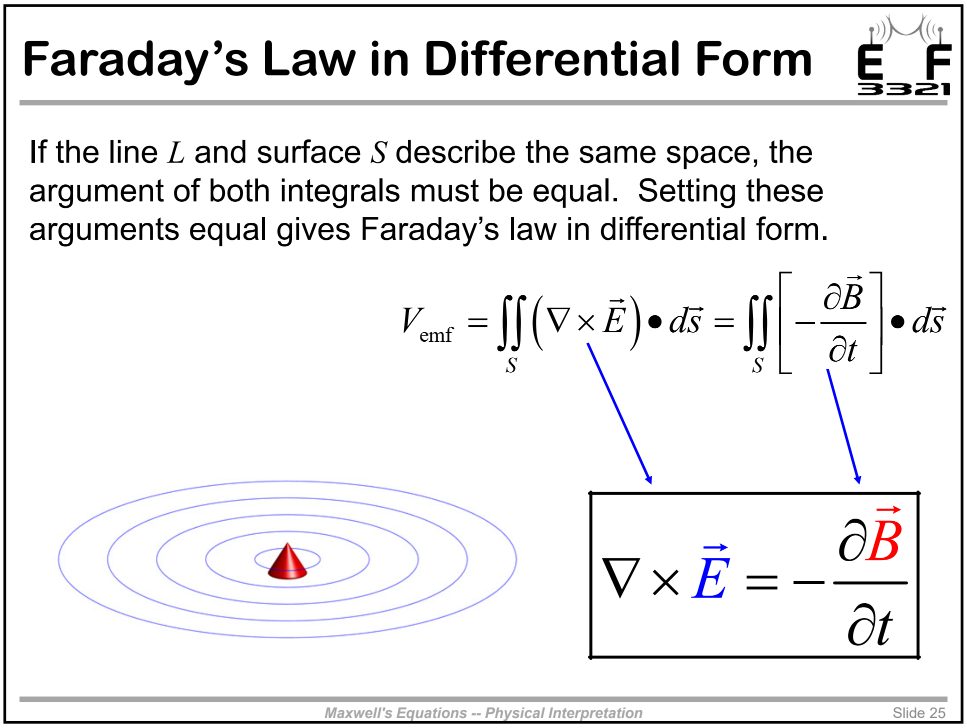
$$\oint_L \vec{A} \bullet d\vec{\ell} = \iint_S \left(\nabla \times \vec{A} \right) \bullet d\vec{s}$$

Applying this to Faraday’s law in integral form gives us

$$V_{\text{emf}} = \oint_L \vec{E} \bullet d\vec{\ell} = -\iint_S \left[\frac{\partial \vec{B}}{\partial t} \right] \bullet d\vec{s}$$

$$\downarrow$$

$$V_{\text{emf}} = \iint_S \left(\nabla \times \vec{E} \right) \bullet d\vec{s} = -\iint_S \left[\frac{\partial \vec{B}}{\partial t} \right] \bullet d\vec{s}$$



Faraday's Law in Differential Form

If the line L and surface S describe the same space, the argument of both integrals must be equal. Setting these arguments equal gives Faraday's law in differential form.

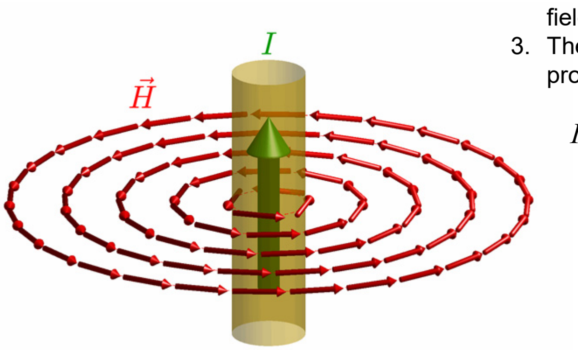
$$V_{\text{emf}} = \iint_S \left(\nabla \times \vec{E}\right) \bullet d\vec{s} = \iint_S \left[-\frac{\partial \vec{B}}{\partial t}\right] \bullet d\vec{s}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Physical Meaning of Maxwell's Equations:

Ampere's Circuit Law

Ampere's Experiment



Observations:

1. There exists a circulating magnetic field H around a conductor carrying a current I.
2. The current I can induce the magnetic field H, or the magnetic field H can induce the current I.
3. The measured current I is in proportion to the circulation of H.

$$I = \oint_L \vec{H} \bullet d\vec{\ell}$$

Three Types of Current

The total current I can be calculated by integrating the flux of the electric current density J through a cross section of the conductor.

$$I = \iint_S \vec{J}_{\text{total}} \bullet d\vec{s}$$

However, recall that there are three types of electric current.

$$\vec{J}_{\text{total}} = \underbrace{\vec{J}_\sigma + \vec{J}_\varepsilon}_{\vec{J}} + \vec{J}_D$$

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t} \equiv \text{displacement current}$$

$$\vec{J} \equiv \text{current due to free charges}$$

Putting this together gives

$$I = \iint_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] \bullet d\vec{s}$$

Ampere's Circuit Law in Integral Form

We have two ways of calculating the total current I in a conductor. Setting these equal gives…

Ampere's Experiment

Simple integration of current density

$$\boxed{I = \oint_L \vec{H} \bullet d\vec{\ell} = \iint_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] \bullet d\vec{s}}$$

Apply Stoke's Theorem

Stoke's theorem allows us to write a closed-contour line integral as a surface integral.

$$\oint_L \vec{A} \bullet d\vec{\ell} = \iint_S \left(\nabla \times \vec{A} \right) \bullet d\vec{s}$$

Applying this to Ampere's law in integral form gives us

$$I = \oint_L \vec{H} \bullet d\vec{\ell} = \iint_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] \bullet d\vec{s}$$

$$\downarrow$$

$$I = \iint_S \left(\nabla \times \vec{H} \right) \bullet d\vec{s} = \iint_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] \bullet d\vec{s}$$

Ampere's Circuit Law in Differential Form

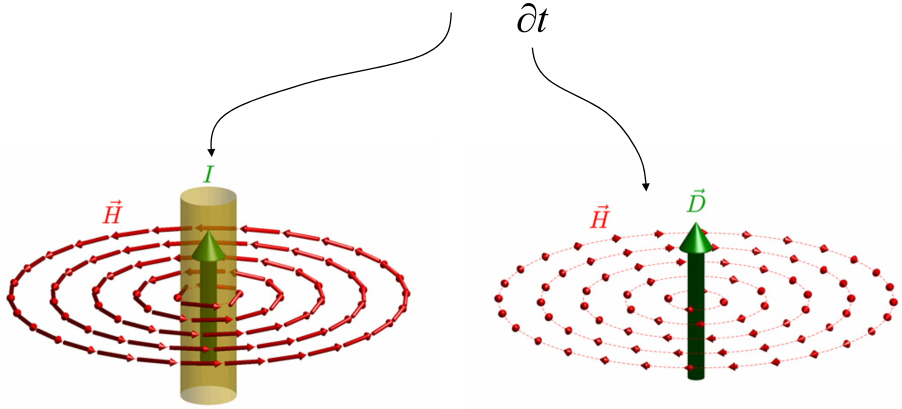
If the line L and surface S describe the same space, the argument of both integrals must be equal. Setting these arguments equal gives Ampere's circuit law in differential form.

$$I = \iint_S \left(\nabla \times \vec{H}\right) \bullet d\vec{s} = \iint_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] \bullet d\vec{s}$$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

Visualization of Ampere's Circuit Law in Differential Form

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$



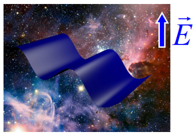
Physical Meaning of Maxwell's Equations:

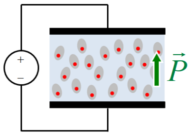
Constitutive Relations

Electric Response of Materials

$$\vec{D} = \underbrace{\varepsilon_0 \vec{E}}_{\text{Vacuum response}} + \underbrace{\vec{P}}_{\text{Material response}}$$

Note: ε_0 is the free space permittivity and multiplies E so that $\varepsilon_0 E$ has the same units as P.





Electric Polarization

In general, the relation between the applied electric field E and the electric polarization P is nonlinear so it can be expressed as a polynomial.

$$P = \underbrace{\varepsilon_0 \chi_{\mathrm{e}}^{(1)} E}_{\text{Linear response}} + \underbrace{\varepsilon_0 \chi_{\mathrm{e}}^{(2)} E^2 + \varepsilon_0 \chi_{\mathrm{e}}^{(3)} E^3 + \cdots}_{\text{Nonlinear response}}$$

These terms are usually ignored. They tend to only become significant when the electric field is very strong.

$\chi_{\mathrm{e}}^{(2)}$ is pronounced "chi two"

$\chi_{\mathrm{e}}^{(2)}$ is pronounced "chi three"

$\vdots$

$$\chi_{\mathrm{e}}^{(n)} \equiv \text{electric susceptibility } (\text{no units})$$

Electric Permittivity & Susceptibility

The permittivity is related to the electric susceptibility through

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_{\mathrm{e}}^{(1)} \vec{E} = \varepsilon_0 \left(1 + \chi_{\mathrm{e}}^{(1)}\right) \vec{E}$$

The constitutive relation can also be written in terms of the relative permittivity ε_{r}.

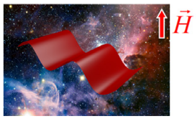
$$\vec{D} = \varepsilon_0 \varepsilon_{\mathrm{r}} \vec{E}$$

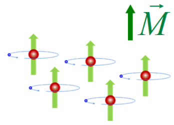
$$\varepsilon_{\mathrm{r}} = \underbrace{1}_{\text{Vacuum response}} + \underbrace{\chi_{\mathrm{e}}^{(1)}}_{\text{Dielectric response}}$$

Magnetic Response of Materials

$$\vec{B} = \underbrace{\mu_0 \vec{H}}_{\text{Vacuum response}} + \underbrace{\vec{M}}_{\text{Material response}}$$

Note: μ_0 is the free space permeability and multiples H so that $\mu_0 H$ has the same units as M.





Magnetic Polarization

In general, the relation between the applied magnetic field H and the magnetic polarization M is nonlinear so it can be expressed as a polynomial.

$$M = \underbrace{\mu_0 \chi_{\mathrm{m}}^{(1)} H}_{\text{Linear response}} + \underbrace{\mu_0 \chi_{\mathrm{m}}^{(2)} H^2 + \mu_0 \chi_{\mathrm{m}}^{(3)} H^3 + \cdots}_{\text{Nonlinear response}}$$

These terms are usually ignored. They tend to only become significant when the electric field is very strong.

$\chi_{\mathrm{m}}^{(2)}$ is pronounced "chi two"

$\chi_{\mathrm{m}}^{(2)}$ is pronounced "chi three"

$\vdots$

$$\chi_{\mathrm{m}}^{(n)} \equiv \text{magnetic susceptibility (no units)}$$

Magnetic Permeability & Susceptibility

The permeability is related to the magnetic susceptibility through

$$\vec{B} = \mu_0\mu_r\vec{E} = \mu_0\vec{H} + \mu_0\chi_m^{(1)}\vec{H} = \mu_0\left(1+\chi_m^{(1)}\right)\vec{H}$$

$$\mu_r = 1 + \chi_m^{(1)}$$

Vacuum response

Magnetic response

Types of Magnetic Materials

- Diamagnetic
  - Negative magnetic susceptibility ($\chi_m < 0$)
  - Tends to oppose an applied magnetic field.
  - All materials are diamagnetic, but usually very week.
  - Copper, silver, gold
- Paramagnetic
  - Small positive susceptibility ($\chi_m > 0$ but small)
  - Material is magnetizable and is attracted to an applied magnetic field.
  - Does not retain magnetization when the external field is removed.
- Ferromagnetic
  - Large positive susceptibility
  - Like paramagnetic, but they retain their magnetism to some degree when the external field is removed.
  - Iron, nickel, cobalt, and some allows.

Anisotropy

The dielectric response of a material arises due to the electric field displacing charges.

Due to structural and bonding effects at the atomic scale, charges are often more easily displaced in some directions than others.

This gives rise to anisotropy where the electric field may experience an entirely different permittivity depending on what direction it is oriented.

$$\vec{D} = \underset{\uparrow \text{ tensor}}{[\varepsilon]}\vec{E}$$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}}_{[\varepsilon]} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

ε_{ij} = how much of E_j contributes to D_i

Types of Anisotropy (1 of 2)

Isotropic Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

This is the typical approximation made in electromagnetics. The permittivity tensor reduces to a scalar.

$$\vec{D} = \varepsilon\vec{E}$$

Uniaxial Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \varepsilon_o & 0 & 0 \\ 0 & \varepsilon_o & 0 \\ 0 & 0 & \varepsilon_e \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

Anisotropic materials are said to be birefringent.

$$\Delta\varepsilon = \varepsilon_e - \varepsilon_o$$

Positive birefringence: $\Delta\varepsilon > 0$

Negative birefringence: $\Delta\varepsilon < 0$

$\varepsilon_o \equiv$ ordinary permittivity

$\varepsilon_e \equiv$ extraordinary permittivity

Biaxial Media

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \varepsilon_a & 0 & 0 \\ 0 & \varepsilon_b & 0 \\ 0 & 0 & \varepsilon_c \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

When orientation is not important, it so convention to order the tensor elements according to $\varepsilon_a < \varepsilon_b < \varepsilon_c$

Types of Anisotropy (2 of 2)

Doubly Anisotropic

$$\vec{D} = [\varepsilon]\vec{E} \quad \text{and} \quad \vec{B} = [\mu]\vec{H}$$

Chiral Materials

$$\begin{bmatrix} D_a \\ D_b \\ D_c \end{bmatrix} = \begin{bmatrix} \varepsilon_a & -j\varepsilon_b & 0 \\ j\varepsilon_b & \varepsilon_a & 0 \\ 0 & 0 & \varepsilon_c \end{bmatrix} \begin{bmatrix} E_a \\ E_b \\ E_c \end{bmatrix}$$

Gyroelectric

$$\begin{bmatrix} B_a \\ B_b \\ B_c \end{bmatrix} = \begin{bmatrix} \mu_a & -j\mu_b & 0 \\ j\mu_b & \mu_a & 0 \\ 0 & 0 & \varepsilon_c \end{bmatrix} \begin{bmatrix} H_a \\ H_b \\ H_c \end{bmatrix}$$

Gryomagnetic

Ordinary and Bi- Materials

	Ordinary Materials	Bi- Materials
Isotropic Materials	**Isotropic Materials** $\vec{D} = \varepsilon\vec{E}$ $\vec{B} = \mu\vec{H}$	**Bi-Isotropic Materials** $\vec{D} = \varepsilon\vec{E} + \xi\vec{H}$ $\vec{B} = \xi\vec{E} + \mu\vec{H}$ $\xi \equiv$ magnetoelectric coupling coefficient
Anisotropic Materials	**Anisotropic Materials** $\vec{D} = [\varepsilon]\vec{E}$ $\vec{B} = [\mu]\vec{H}$	**Bi-Anisotropic Materials** $\vec{D} = [\varepsilon]\vec{E} + [\xi]\vec{H}$ $\vec{B} = [\xi]^{\mathrm{T}}\vec{E} + [\mu]\vec{H}$