Electrostatics: Example – Uniform Circular Plate Charge

What is the total charge $Q_{\text{total}}$?

1. Draw the problem.
2. Choose a coordinate system. Cylindrical
3. Write general equation.
   $Q_{\text{total}} = \iiint \rho_e \, dz$
4. Write expressions for each term.
   $\rho_e = \rho_z = \rho_s \rho d\phi$
5. Choose limits of integration.
   $Q_{\text{total}} = \int_{\phi=0}^{\phi=\pi} \rho_s \rho d\phi d\phi$
Total Charge

What is the total charge $Q_{\text{Total}}$?

6. Solve the integral.

$$Q_{\text{Total}} = \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=R} \rho \, d\rho \, d\phi$$

$$= \rho \int_{\rho=0}^{\rho=R} \rho \, d\rho \int_{\phi=0}^{\phi=2\pi} d\phi$$

$$= \frac{\rho R^2}{2} \int_{\phi=0}^{\phi=2\pi} d\phi$$

$$= \frac{\rho R^2}{2} \cdot 2\pi$$

$$Q_{\text{Total}} = \pi R^2 \rho$$

7. Interpret the result.

$Q_{\text{Total}} = \rho S$ for uniform charge density

Total Field

What is the total field $\vec{D}$?

1. Draw the problem.

2. Choose a coordinate system. Cylindrical

3. Write general equation.

$$\vec{D}_{\text{Total}} = \int_{z} \int_{R} \vec{D}_{\rho,\phi,\theta} \rho d\rho d\phi$$

4. Write expressions for each term.

$$\rho = \rho_s \quad R = (0,0,z) \rightarrow \rho, \phi, 0 = (-\rho, 0, z)$$

$$ds = \rho d\rho d\phi \quad \vec{k} = \sqrt{\rho^2 + z^2}$$

5. Choose limits of integration.

$$\vec{D}_{\text{Total}} = \int_{z} \int_{R} \frac{\rho_s}{4\pi \epsilon_0} \frac{1}{\sqrt{\rho^2 + z^2}} \rho d\rho d\phi$$
What is the total field \( \vec{D} \)?

6. Solve the integral.

\[
\vec{D}_{\text{total}} = \frac{\rho_s}{4\pi} \int_0^{\pi} \int_{\rho_s}^{R} \frac{\rho d\phi}{(\rho^2 + z^2)^{3/2}} r d\rho
\]

Due to symmetry, we can ignore the \( \rho \) component.

\[
\vec{D}_{\text{total}} = \frac{\rho_s}{4\pi} \int_0^{\pi} \int_{\rho_s}^{R} \frac{\rho d\phi}{(\rho^2 + z^2)^{3/2}} r d\rho d\phi
\]

\[
= \frac{\rho_s}{4\pi} \int_0^{\pi} \left[ \frac{1}{2(\rho^2 + z^2)^{1/2}} \right]_{\rho_s}^{R} d\phi
\]

\[
= \frac{\rho_s}{4\pi} \int_0^{\pi} \left[ (\rho^2 + z^2)^{-1/2} - (\rho_s^2 + z^2)^{-1/2} \right] d\phi
\]

\[
= \frac{\rho_s}{2} \int_0^{2\pi} \left[ (\rho^2 + z^2)^{-1/2} \right] d\phi
\]

\[
= \frac{\rho_s}{2} \int_0^{2\pi} \frac{1}{(\rho^2 + z^2)^{1/2}} d\phi
\]

\[
= \frac{\rho_s}{2} \left[ \frac{\phi}{\sqrt{\rho^2 + z^2}} \right]_0^{2\pi}
\]

\[
= \frac{\rho_s}{2} \left[ 2\pi - 0 \right]
\]

\[
= \pi \rho_s
\]

Our integral in terms of \( \rho \) is then

\[
\vec{D}_{\text{total}} = \pi \rho_s \frac{\rho d\phi}{(\rho^2 + z^2)^{3/2}} r d\rho
\]

Total Field

What is the total field \( \vec{D} \)?

6. Solve the integral.

\[
\vec{D}_{\text{total}} = \frac{\rho_s}{2} \int_0^{2\pi} \left( (\rho^2 + z^2)^{-1/2} \right) r d\rho d\phi
\]

Let \( v = \rho^2 + z^2 \)

Then \( dv = 2\rho d\rho \)

\[
d\rho = \frac{dv}{2\rho}
\]

\[
\rho_s = 0 \rightarrow v_s = 0^2 + z^2 = z^2
\]

\[
\rho_s = R \rightarrow v_s = R^2 + z^2
\]

Our integral in terms of \( v \) is then

\[
\vec{D}_{\text{total}} = \frac{\rho_s}{2} \int_0^{2\pi} v^{1/2} \frac{dv}{2\rho}
\]

\[
= \frac{\rho_s}{4} \int_0^{2\pi} v^{1/2} dv
\]

\[
= \frac{\rho_s}{4} \left[ \frac{v^{3/2}}{3/2} \right]_0^{2\pi}
\]

\[
= \frac{\rho_s}{4} \left[ \frac{2\pi}{3/2} \right]
\]

\[
= \frac{\rho_s}{4} \left[ \frac{4\pi}{3} \right]
\]

\[
= \frac{\rho_s}{3} \pi
\]
Total Field

What is the total field $\vec{D}$?

6. Solve the integral.

$$\vec{D}_{\text{total}} = \frac{\rho_z}{4} \hat{a}_z \int_{\text{cylinder}} v_r^2 \; dv_r$$

$$= \frac{\rho_z}{4} \hat{a}_z \left[ \frac{v_r^2}{2} \right]_{\text{cylinder}}$$

$$= \frac{\rho_z}{2} \hat{a}_z \left( \frac{1}{R^2 + z^2} \right)$$

$$= \frac{\rho_z}{2} \left( \frac{z}{\sqrt{R^2 + z^2}} \right) \hat{a}_z$$

$$\vec{D}_{\text{total}} = \frac{\rho_z}{2} \left( \frac{z}{\sqrt{R^2 + z^2}} \right) \hat{a}_z$$