Electrostatics:
Energy in Electrostatic Fields

EE3321
Electromagnetic Field Theory

Outline

• Energy in terms of potential
• Energy in terms of the field
• Power and energy in conductors
Recall the relation between potential difference, work, and charge.

\[ V_{AB} = V_B - V_A = \frac{W}{Q} \]

Therefore, the work it takes to move charge \( Q \) from A to B is

\[ W = QV_{AB} \]
An ensemble of charges contains energy because the charges are putting a force on each other and so they have the potential to do work.

We will calculate how much energy the ensemble contains by calculating how much energy it took to assemble it.

No other charges are present, so placing $Q_1$ at $P_1$ takes no work.

$$W_1 = 0$$
Point Charge #2

Placing $Q_2$ at $P_2$ takes work because charge $Q_1$ is present.

$$W_2 = Q_2 V_{21}$$

Point Charge #3

Placing $Q_3$ at $P_3$ takes work because charges $Q_1$ and $Q_2$ are present.

$$W_3 = Q_3 V_{31} + Q_3 V_{32}$$
**Total Work So Far**

The total work placing all three charges is

\[ W = W_1 + W_2 + W_3 = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \]

**Assembly in Reverse Order**

If we had placed the charges in the reverse order,

\[ W = W_3 + W_2 + W_1 = 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13}) \]
Add Both Approaches

\[ W = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \]

Equation obtained by placing \( Q_1 \), then \( Q_2 \), and then \( Q_3 \).

\[ W = 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13}) \]

Equation obtained by placing \( Q_3 \), then \( Q_2 \), and then \( Q_1 \).

\[ 2W = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) + 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13}) \]

Add the two equations above.

\[ 2W = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32}) \]

Total potentials \( \rightarrow V_1 \quad V_2 \quad V_3 \)

\[ 2W = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \]

Final Expression

\[ 2W = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \]

\[ W = \frac{1}{2} Q_1 V_1 + \frac{1}{2} Q_2 V_2 + \frac{1}{2} Q_3 V_3 \]

Solve for \( W \).

It is straightforward to generalize this for any number of charges.

\[ W = \frac{1}{2} \sum_{i=1}^{N} Q_i V_i \quad (\text{joules}) \]
### Energy in Charge Distributions

<table>
<thead>
<tr>
<th>Point Charge</th>
<th>Line Charge</th>
<th>Sheet Charge</th>
<th>Volume Charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$ (C)</td>
<td>$\rho_v$ (C/m)</td>
<td>$\rho_s$ (C/m$^2$)</td>
<td>$\rho_v$ (C/m$^3$)</td>
</tr>
<tr>
<td>Total Charge $Q_{\text{total}} = \sum_{i=1}^{N} Q_i$</td>
<td>Total Charge $Q_{\text{total}} = \int \rho_v d\ell \geq \rho_s L$</td>
<td>Total Charge $Q_{\text{total}} = \int \rho_s ds \geq \rho_s S$</td>
<td>Total Charge $Q_{\text{total}} = \iiint \rho_v dv = \rho_v V$</td>
</tr>
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</tr>
</tbody>
</table>

#### Energy in Terms of the Field

Electrostatics - Energy
Derivation (1 of 5)

The energy in a volume charge is

\[ W = \frac{1}{2} \iiint_V \rho_v V dv \]

Recall from Maxwell’s equations that \( \rho_v = \nabla \cdot \vec{D} \).

\[ W = \frac{1}{2} \iiint_V (\nabla \cdot \vec{D}) V dv \]

Recall the product rule for divergence \( \nabla \cdot (f \vec{A}) = f (\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla f \)

\[ \nabla \cdot (V \vec{D}) = V (\nabla \cdot \vec{D}) + \vec{D} \cdot \nabla V \]

\[ (\nabla \cdot \vec{D}) V = \nabla \cdot (V \vec{D}) - \vec{D} \cdot \nabla V \]

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Derivation (2 of 5)

Apply the product rule for our equation for work.

\[ W = \frac{1}{2} \iiint_V (\nabla \cdot \vec{D}) V dv \]

\[ = \frac{1}{2} \iiint_V \left[ \nabla \cdot (V \vec{D}) - \vec{D} \cdot \nabla V \right] dv \]

\[ = \frac{1}{2} \iiint_V \left[ \nabla \cdot (V \vec{D}) \right] dv - \frac{1}{2} \iiint_V [\vec{D} \cdot \nabla V] dv \]
Recall the divergence theorem
\[
\iint_S \vec{F} \cdot d\vec{s} = \iiint_V \left( \nabla \cdot \vec{F} \right) \, dv
\]
Apply this to our equation for work.
\[
W = \frac{1}{2} \iiint_V \left[ \nabla \cdot \left( V \vec{D} \right) \right] \, dv - \frac{1}{2} \iiint_V \left[ \vec{D} \cdot \nabla V \right] \, dv
\]
Overall
\[
\frac{1}{2} \iint_S \left( V \vec{D} \right) \cdot d\vec{s} - \frac{1}{2} \iiint_V \left[ \vec{D} \cdot \nabla V \right] \, dv
\]

Look more closely at the surface integral.
\[
W = \frac{1}{2} \iint_S \left( V \vec{D} \right) \cdot d\vec{s} - \frac{1}{2} \iiint_V \left[ \vec{D} \cdot \nabla V \right] \, dv
\]
Overall
\[
\frac{1}{2} \iint_S \left( V \vec{D} \right) \cdot d\vec{s} - \frac{1}{2} \iiint_V \left[ \vec{D} \cdot \nabla V \right] \, dv
\]
We are free to choose whatever surface $S$ we wish.
As we enlarge the surface out to infinity, the surface integral becomes negligible relative to the volume integral.
Derivation (5 of 5)

Our equation for work is now

\[ W = -\frac{1}{2} \iiint_{V} [\mathbf{D} \cdot \nabla V] \, dv \]

Associate the negative sign with \( \nabla V \).

\[ W = \frac{1}{2} \iiint_{V} [\mathbf{D} \cdot (-\nabla V)] \, dv \]

This is the electric field intensity \( \mathbf{E} \).

This is the general equation for energy stored in the electrostatic field.
It is valid for anisotropic and inhomogeneous media.

Electrostatic Energy in LHI Media

The more common expression for energy in the electrostatic field is for the special case of linear, homogeneous, and isotropic (LHI) media.

In isotropic media we have \( \mathbf{D} = \varepsilon \mathbf{E} \).

\[ W = \frac{1}{2} \iiint_{V} (\mathbf{D} \cdot \mathbf{E}) \, dv \]

\[ = \frac{1}{2} \iiint_{V} (\varepsilon \mathbf{E} \cdot \mathbf{E}) \, dv \]

This is the general equation for energy stored in the electrostatic field.
It is valid for anisotropic and inhomogeneous media.

\[ W = \frac{1}{2} \iiint_{V} \varepsilon |\mathbf{E}|^2 \, dv \]

Simpler equation that is only valid in LHI media.
Electrostatic Energy Density

Observe what we have been integrating to get total energy.

\[ W = \iiint \left( \frac{1}{2} \vec{D} \cdot \vec{E} \right) dv \]
\[ W = \iiint \left( \frac{1}{2} \varepsilon |\vec{E}|^2 \right) dv \]

These expressions must be energy density \( w \).

We can now think of calculating total energy by integrating the energy density \( w \).

\[ W = \iiint w dv \]

\[ w = \begin{cases} 
\frac{1}{2} \vec{D} \cdot \vec{E} & \text{General case} \\
\frac{1}{2} \varepsilon |\vec{E}|^2 & \text{LHI media}
\end{cases} \]

Power & Energy in Conductors
Joule’s Law

Joule’s law states that

\[ P = \iiint (\vec{E} \cdot \vec{J}) \, dv \]

This is equivalent to \( P = VI \) in circuit theory.

From this, we can extract the energy density in a conductor.

\[ w = \vec{E} \cdot \vec{J} \]

Applying Ohm’s law for electromagnetics \( \vec{J} = \sigma \vec{E} \) gives

\[ w = \vec{E} \cdot \vec{J} \]

\[ = \vec{E} \cdot \sigma \vec{E} \]

\[ = \sigma |\vec{E}|^2 \]

\[ P = \iiint \sigma |\vec{E}|^2 \, dv \]

Most common form.