Electrostatics: Capacitor Examples

EE3321
Electromagnetic Field Theory

Outline

• Parallel plate capacitor
• How big is a Farad?
• Coaxial capacitor
• RG-59 coax
• Inhomogeneous capacitor
Example #1: Parallel Plate Capacitor

Step 1 – Choose a convenient coordinate system. 

Cartesian
Example 1

Step 1 – Choose a convenient coordinate system.

**Cartesian**

Step 2 – Let the plates carry charges \(+Q\) and \(-Q\).

Step 3 – Calculate \(\vec{D}\) using Gauss’ law.

Recall the field around an infinite plate.

\[
\vec{D} = \frac{\rho_s}{2} \hat{a}_n
\]

Field below the top plate,

\[
\vec{D}_{\text{top}} = -\frac{\rho_s}{2} \hat{a}_z
\]
Example 1

Step 3 – Calculate $\vec{D}$ using Gauss’ law.

Recall the field around an infinite plate.

$$\vec{D} = \frac{\rho_s}{2} \hat{a}_n$$

Field below the top plate,

$$\vec{D}_{\text{top}} = -\frac{\rho_s}{2} \hat{a}_n$$

Field above the bottom plate,

$$\vec{D}_{\text{bot}} = -\frac{\rho_s}{2} \hat{a}_z$$

When both plates are considered

$$\vec{D} = \vec{D}_{\text{top}} + \vec{D}_{\text{bot}} = -\rho_s \hat{a}_z$$
Example 1

Step 3 – Calculate $\vec{D}$ using Gauss’ law.

The surface charge density is

$$\rho_s = \frac{Q}{S}$$

The final expression for $\vec{D}$ is

$$\vec{D} = -\rho_s \hat{a}_z = -\frac{Q}{S} \hat{a}_z$$

Example 1

Step 4 – Calculate $\vec{E}$.

We calculate $\vec{E}$ from the constitutive relation.

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = -\frac{Q}{\varepsilon S} \hat{a}_z$$
Example 1

Step 5 – Calculate $V_0$.

Given $\vec{E}$, we calculate $V_0$ by integrating from the bottom plate to the top plate.

\[ V_0 = -\int_L \vec{E} \cdot d\vec{r} \]

\[ V_0 = -\int_0^d \left( -\frac{Q}{\varepsilon S} \hat{a}_z \right) \cdot d\hat{a}_z \]

\[ V_0 = \frac{Q}{\varepsilon S} \int_0^d dz \]

\[ V_0 = \frac{Qd}{\varepsilon S} \]

Example 1

Step 6 – Calculate capacitance $C$.

\[ C = \frac{|Q|}{|V_0|} = \frac{|Q|}{|Qd/\varepsilon S|} = \frac{Qd}{\varepsilon S} = \frac{\varepsilon S}{d} \]

The final answer is

\[ C = \frac{\varepsilon S}{d} \]

Self-check – $C$ should not be a function of $Q$ or $V_0$. 
Example #2: How Big is a Farad?

Suppose our plates are 10 m by 20 m and the gap between the plates is 1 mm.

\[
C = \frac{\varepsilon S}{d} = \frac{\varepsilon_0 \varepsilon_r LW}{d} = \frac{(8.854 \times 10^{-12} \text{ F/m})(1.0)(10 \text{ m})(20 \text{ m})}{(0.001 \text{ m})} \\
= 1.78 \times 10^{-6} \text{ F} \\
= 1.78 \mu\text{F}
\]

Our capacitor is physically very large, yet the capacitance is very small.
The Farad is a HUGE unit!!!
Example #3: Coaxial Capacitor

Step 1 – Choose a convenient coordinate system.

Cylindrical \((\rho, \phi, z)\)
Example #3 – Coaxial Capacitor

Step 1 – Choose a convenient coordinate system.

Cylindrical \((\rho, \phi, z)\)

Step 2 – Let the plates carry charges \(+Q\) and \(-Q\).

Step 3 – Calculate \(\vec{D}\) using Gauss' law.

We define a Gaussian surface with radius \(\rho\) to be inside of the dielectric.

\[ Q = \iiint_{S} \vec{D} \cdot d\vec{s} \]

From our boundary conditions, we know the electric field will be normal at the interfaces to the metal.
Example #3 – Coaxial Capacitor

Step 3 – Calculate $\vec{D}$ using Gauss’ law.

The only field configuration that makes sense considering the boundary conditions is when the field is purely radially directed.

$$\vec{D} = D_\rho (\rho, \phi, z) \hat{\rho}$$

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Gauss' law becomes

$$Q = \iint_{S_{2\pi}} \left( D_\rho \hat{\rho} \right) \cdot \left( \rho d\phi dz \hat{\rho} + d \rho dz \hat{\theta} + \rho d\phi dz \hat{z} \right)$$

$$= \iiint_{V} D_\rho \rho d\phi dz$$

$$= D_\rho \rho \left( \int_{0}^{2\pi} d\phi \right) \left( \int_{0}^{L} dz \right)$$

$$= D_\rho \rho \left( 2\pi \right) \left( L \right)$$

$$= 2\pi D_\rho \rho \left( \int_{0}^{L} dz \right) = 2\pi D_\rho \rho L$$
Step 3 – Calculate $\vec{D}$ using Gauss’ law.

Solving for $\vec{D}$ gives

$$D_\rho(\rho, \phi, z) = \frac{Q}{2\pi\rho L}$$

$D$ only depends on $\rho$.

$$\vec{D}(\rho) = \frac{Q}{2\pi\rho L} \hat{a}_\rho$$

Step 4 – Calculate $\vec{E}$.

We calculate $\vec{E}$ from the constitutive relation.

$$\vec{E}(\rho) = \frac{\vec{D}}{\varepsilon} = \frac{Q}{2\pi\varepsilon\rho L} \hat{a}_\rho$$

Step 5 – Calculate $V_0$.

Given $\vec{E}$, we calculate $V_0$ by integrating from the inner conductor to the outer conductor.

$$V_0 = -\int_L \vec{E} \cdot d\vec{r}$$

$$= -\int_a^b \left( \frac{Q}{2\pi\varepsilon\rho L} \hat{a}_\rho \right) \cdot (d\rho \hat{a}_\rho)$$

$$= -\int_a^b \frac{Q}{2\pi\varepsilon\rho L} d\rho$$

$$= -\frac{Q}{2\pi\varepsilon L} \left[ \ln \frac{\rho}{a} \right]^b_a$$

$$= -\frac{Q}{2\pi\varepsilon L} \left( \ln \frac{b}{a} \right)$$
Example #3 – Coaxial Capacitor

Step 5 – Calculate $V_0$.

Continued…

$$V_0 = -\frac{Q}{2\pi \varepsilon L} (\ln \rho_b^\circ)$$

$$= -\frac{Q}{2\pi \varepsilon L} (\ln b - \ln a)$$

$$= \frac{Q}{2\pi \varepsilon L} (\ln a - \ln b)$$

$$= \frac{Q}{2\pi \varepsilon L} \ln \left(\frac{a}{b}\right)$$

Example #3 – Coaxial Capacitor

Step 6 – Calculate capacitance $C$.

$$C = \frac{|Q|}{|V_0|}$$

$$= \frac{Q}{2\pi \varepsilon L} \ln \left(\frac{a}{b}\right)$$

$$C = \frac{2\pi \varepsilon L}{\ln \left(\frac{a}{b}\right)}$$

* Self Check – $C$ is not a function of $Q$ or $V_0$. 
Example #3 – Coaxial Capacitor

Distributed Capacitance

We would like to specify the capacitance without knowledge of $L$.

We do this using the distributed capacitance, which capacitance per unit length.

$$\frac{C}{L} = \frac{2\pi \varepsilon}{\ln \left( \frac{a}{b} \right)}$$

Example #4:

RG-59 Coax
Example #4 – RG-59 Coax

A standard RG-59 coax has

- Inner conductor diameter: 0.81 mm (20 AWG)
- Outer conductor diameter: 3.66 mm
- Dielectric constant: 2.1
- Specified capacitance: 86.9 pF/m

\[
\frac{C}{L} = \frac{2\pi \varepsilon_0 \varepsilon_r}{\ln(a/b)}
\]

\[
= \frac{2\pi \left(8.854 \times 10^{-12} \text{ F/m}\right)(2.1)}{\ln(3.66 \text{ mm}/0.81 \text{ mm})}
\]

\[
= 7.746 \times 10^{-11} \text{ F/m} = 77.46 \text{ pF/m}
\]

Example #5:
Inhomogeneous Capacitor
Suppose we have an inhomogeneous capacitor.

We split the inhomogeneous capacitor into a combination of homogeneous capacitors.
Example #5 – Inhomogeneous Capacitor

Calculate each homogeneous capacitor independently.

\[ C_1 = \frac{\varepsilon_1 S_1}{d_1} = \frac{\varepsilon_1 t w_1}{d_1} \]

\[ C_2 = \frac{\varepsilon_2 S_2}{d_1} = \frac{\varepsilon_2 t w_2}{d_1} \]
Example #5 – Inhomogeneous Capacitor

Calculate each homogeneous capacitor independently.

\[ C_1 = \frac{\varepsilon_1 S_1}{d_1} = \frac{\varepsilon_1 t w_1}{d_1} \]

\[ C_2 = \frac{\varepsilon_2 S_2}{d_1} = \frac{\varepsilon_2 t w_2}{d_1} \]

\[ C_3 = \frac{\varepsilon_3 S_3}{d_2} = \frac{\varepsilon_3 t (w_1 + w_2)}{d_2} \]

Example #5 – Inhomogeneous Capacitor

We view our capacitor as a series/parallel combination of capacitors.

The equivalent capacitance is

\[ C_{eq} = \left( C_1 + C_2 \right) || C_3 \]

\[ = \left( \frac{\varepsilon_1 t w_1}{d_1} + \frac{\varepsilon_2 t w_2}{d_1} \right) \frac{\varepsilon_3 t (w_1 + w_2)}{d_2} \]

\[ C_{eq} = \frac{t \varepsilon_3 (\varepsilon_1 w_1 + \varepsilon_2 w_2)(w_1 + w_2)}{d_2 (\varepsilon_1 w_1 + \varepsilon_2 w_2) + \varepsilon_3 d_1 (w_1 + w_2)} \]