Magnetostatics: 
Forces on Charged Particles

EE3321
Electromagnetic Field Theory

Outline

• Forces due to magnetic fields
• Example #1 – velocity filter
• Example #2 – deriving particle position
Forces Due to Magnetic Fields

Mechanisms to Produce Force

- Force on a moving charged particle
- Force on a current element
- Force between two current element
**Lorentz Force Equation**

\[ \vec{F} = Q \vec{E} + Q \vec{u} \times \vec{B} \]

- **Electrostatic Force**
  \[ \vec{F}_e = Q \vec{E} \]
  - Force \( \vec{F}_e \) is in same direction as \( \vec{E} \).
  - Force is independent of velocity.
  - Can change velocity of the charge.
  - \( \vec{F}_e \) is usually much greater than \( \vec{F}_m \) except at very high velocity.

- **Magnetostatic Force**
  \[ \vec{F}_m = Q \vec{u} \times \vec{B} \]
  - Force \( \vec{F}_m \) is perpendicular to \( \vec{B} \) and velocity \( \vec{u} \).
  - Force is only exerted on moving charges.
  - Cannot change kinetic energy of the charge (i.e. \( \vec{F}_m \cdot d\vec{l} = 0 \)).
  - Magnetic field can only change the direction of velocity.

**Recall F = ma**

\[ \vec{F} = m \frac{d\vec{u}}{dt} = Q \vec{E} + Q \vec{u} \times \vec{B} \]

This equation is used to calculate total force on a charge as well as its position, velocity, and acceleration.
Example #1: Velocity Filter
Example #1 – Velocity Filter

Suppose we wish to create a “bandpass” velocity filter where the system only passes charges moving with velocity 4 m/s and deflects all others.

Without any applied field, charges flow in a straight line and charges of all velocities are passed by the aperture.
Example #1 – Velocity Filter

Now we apply an electric field in the vertical direction.

Let this be \( \vec{E} = -(20 \text{ V/m}) \hat{a}_z \).

Example #1 – Velocity Filter

All charges experience the same force in the downward direction.

Particles spread according to velocity due to inertia, not due to a velocity dependent force.
Example #1 – Velocity Filter

We wish to calculate the magnetic field who’s force will exactly oppose the force due to the electric field at the design velocity.

Let this be $\vec{B} = -B_0 \hat{a}_y$.

With just the magnetic field applied, all charges experience a force in the upward direction, but that force is proportional to the velocity.

Deflection appears more uniform because the inertia and velocity dependent force counteract each other.
Example #1 – Velocity Filter

The net force should be zero at the design velocity.

\[ 0 = \vec{Q}\vec{E} + \vec{Q}\vec{u} \times \vec{B} \]

\[ = \vec{E} + \vec{u} \times \vec{B} = -(20 \text{ V/m})\hat{a}_x + (4 \text{ m/s})\hat{a}_z \times (-B_0)\hat{a}_y = -20\hat{a}_x + 4B_0\hat{a}_z \]

\[ B_0 = 5 \text{ Wb/m}^2 \]

Example #1 – Velocity Filter

The forces due to the fields counteract each other such that the charges moving at the design velocity experience zero net force and are not deflected.
Example #2: Deriving Particle Position

A particle with charge 2.0 mC and mass 8 mg is moving at a velocity of 10 m/s in the positive \( x \) direction in the presence of a static magnetic field of \( B = 4\hat{z} \) Wb/m\(^2\). If the particle is at the origin at time \( t = 0 \), what is the particle’s position at \( t = 3 \) s?

**Solution**

Write the Lorentz force equation and include \( d\vec{u}/dt \).

\[
\vec{F} = m \frac{d\vec{u}}{dt} = Q \left( \vec{E} + \vec{u} \times \vec{B} \right)
\]

Solve this for \( d\vec{u}/dt \).

\[
\frac{d\vec{u}}{dt} = \frac{Q}{m} \left( \vec{E} + \vec{u} \times \vec{B} \right)
\]
Example #2 – Particle Position

Expand the cross product.

\[ \frac{du}{dt} = \frac{Q}{m} \left( \vec{E} + \vec{u} \times \vec{B} \right) \]

\[ \frac{d}{dt} \left( u \hat{a}_x + u \hat{a}_y + u \hat{a}_z \right) = \frac{Q}{m} \left[ \left( E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z \right) \hat{a}_x + \left( u B_y - u B_z \right) \hat{a}_y + \left( u B_z - u B_y \right) \hat{a}_z \right] \]

\[ \frac{du_x}{dt} \hat{a}_x + \frac{du_y}{dt} \hat{a}_y + \frac{du_z}{dt} \hat{a}_z = \frac{Q}{m} \left( E_x + u B_y - u B_z \right) \hat{a}_x + \frac{Q}{m} \left( E_y + u B_z - u B_y \right) \hat{a}_y + \frac{Q}{m} \left( E_z + u B_z - u B_y \right) \hat{a}_z \]

We can extract three scalar equations from this one vector equation.

\[ \frac{du_x}{dt} = \frac{Q}{m} \left( E_x + u B_y - u B_z \right) \]
Example #2 – Particle Position

Expand the cross product.

\[ \frac{d\vec{u}}{dt} = \frac{Q}{m} (\vec{E} + \vec{u} \times \vec{B}) \]

\[ \frac{d}{dt}(u_x \dot{a}_x + u_y \dot{a}_y + u_z \dot{a}_z) = \frac{Q}{m} \left[ (E_x \dot{a}_x + E_y \dot{a}_y + E_z \dot{a}_z) + (u_x B_x - u_y B_y) \dot{a}_y + (u_y B_y - u_z B_z) \dot{a}_z + (u_z B_z - u_x B_x) \dot{a}_x \right] \]

\[ \frac{du_x}{dt} \dot{a}_x + \frac{du_y}{dt} \dot{a}_y + \frac{du_z}{dt} \dot{a}_z = \frac{Q}{m} (E_x + u_x B_x - u_z B_z) \dot{a}_y + \frac{Q}{m} (E_y + u_y B_y - u_z B_z) \dot{a}_z + \frac{Q}{m} (E_z + u_z B_z - u_x B_x) \dot{a}_x \]

We can extract three scalar equations from this one vector equation.

\[ \frac{du_x}{dt} = \frac{Q}{m} (E_x + u_x B_x - u_z B_z) \]

\[ \frac{du_y}{dt} = \frac{Q}{m} (E_y + u_y B_y - u_z B_z) \]

\[ \frac{du_z}{dt} = \frac{Q}{m} (E_z + u_z B_z - u_x B_x) \]
Example #2 – Particle Position

Substitute our known values into these equations to simplify them.

\[
\frac{du_x(t)}{dt} = \frac{Q}{m}[E_x + u_x(t)B_y - u_x(t)B_z] = \left(\frac{2 \times 10^{-3}}{8 \times 10^{-3}} \text{ C} \right) \left[0 + u_x(t)(4 \text{ Wb/m}^2) - u_x(t) \cdot 0 \right] = u_x(t)
\]

\[
\frac{du_y(t)}{dt} = \frac{Q}{m}[E_y + u_y(t)B_x - u_y(t)B_z] = \left(\frac{2 \times 10^{-3}}{8 \times 10^{-3}} \text{ C} \right) \left[0 + u_y(t)(0 - u_y(t))(4 \text{ Wb/m}^2) \right] = -u_y(t)
\]

\[
\frac{du_z(t)}{dt} = \frac{Q}{m}[E_z + u_z(t)B_y - u_z(t)B_z] = \left(\frac{2 \times 10^{-3}}{8 \times 10^{-3}} \text{ C} \right) \left[0 + u_z(t) \cdot 0 - u_z(t) \cdot 0 \right] = 0
\]

Write the differential equation for the \( x \) component of velocity.

\[
\frac{du_x(t)}{dt} = u_x(t)
\]

Differentiate with respect to time.

\[
\frac{d}{dt} \left[ \frac{du_x(t)}{dt} \right] = \frac{du_x(t)}{dt}
\]

We can replace \( du_y/dt \) using the differential equation for the \( y \) component.

\[
\frac{d}{dt} \left[ \frac{du_y(t)}{dt} \right] = -u_y(t)
\]

\[
\frac{d}{dt} \left[ \frac{du_z(t)}{dt} \right] = -u_z(t)
\]

Put differential equation in standard form. Solve the differential equation.

\[
\frac{d^2u_x(t)}{dt^2} + u_x(t) = 0 \quad \rightarrow \quad u_x(t) = A \cos t + B \sin t
\]
Example #2 – Particle Position

Substitute the solution back into the original differential equation for the $x$ component to get the solution for $u_x(t)$.
\[
\frac{du_x}{dt} = u_x(t)
\]
\[
\frac{d}{dt}(A \cos t + B \sin t) = u_x(t)
\]
\[-A \sin t + B \cos t = u_x(t)
\]

Write the differential equation for the $z$ component of velocity.
\[
\frac{du_z}{dt} = 0
\]
This is solved by integrating.
\[
u_z(t) = C
\]

Example #2 – Particle Position

Apply the boundary conditions to calculate the constants $A$, $B$, and $C$.

At time $t = 0$, $\vec{u}(0) = (10 \text{ m/s})\hat{a}_x$.
\[
u_x(t) = A \cos t + B \sin t \quad u_x(0) = 10 \text{ m/s} = A \cos 0 + B \sin 0 = A
\]
\[
A = 10 \text{ m/s}
\]
\[
u_y(t) = -A \sin t + B \cos t \quad u_y(0) = 0 = -A \sin 0 + B \cos 0 = B
\]
\[
B = 0
\]
\[
u_z(t) = C \quad u_z(0) = 0 = C
\]
\[
C = 0
\]

Our equations for the components of velocity are now
\[
u_x(t) = 10 \cos t
\]
\[
u_y(t) = -10 \sin t
\]
\[
u_z(t) = 0
\]
Example #2 – Particle Position

Position as a function of time is calculated by integrating the expression for velocity.

\[ \ddot{u}(t) = 10 \cos(t) \hat{a}_x - 10 \sin(t) \hat{a}_y \]

\[ \frac{d\mathbf{r}(t)}{dt} = \ddot{u}(t) = 10 \cos(t) \hat{a}_x - 10 \sin(t) \hat{a}_y \]

\[ \mathbf{r}(t) = \left[ 10 \sin(t) + A \right] \hat{a}_x + \left[ 10 \cos(t) + B \right] \hat{a}_y \]

The constants \( A \) and \( B \) are determined by applying the initial condition.

\[ \mathbf{r}(0) = 0 = \left[ 10 \sin(0) + A \right] \hat{a}_x + \left[ 10 \cos(0) + B \right] \hat{a}_y = A \hat{a}_x + (10 + B) \hat{a}_y \]

\[ A = 0 \quad B = -10 \]

The final expression for position is

\[ \mathbf{r}(t) = 10 \sin(t) \hat{a}_x + 10 \left[ \cos(t) - 1 \right] \hat{a}_y \]

We can finally evaluate the position at \( t = 3 \) s.

\[ \mathbf{r}(t) = 10 \sin(3) \hat{a}_x + 10 \left[ \cos(3) - 1 \right] \hat{a}_y \]

\[ \mathbf{r}(3) = 10 \sin(3) \hat{a}_x + 10 \left[ \cos(3) - 1 \right] \hat{a}_y \]

\[ \mathbf{r}(3) = 1.41 \hat{a}_x - 19.90 \hat{a}_y \]
Example #2 – Particle Position

\[ \mathbf{\ddot{r}}(t) = 10 \sin(t) \hat{a}_x + 10 \left[ \cos(t) - 1 \right] \hat{a}_y \quad t \geq 0 \]

\( t = -2.20 \text{ seconds} \)