EE 4347
Applied Electromagnetics

Topic 3d

Waves in Lossy Dielectrics

Lecture Outline

• Complex Wave Parameters
• Visualization of EM Waves
• Complex Wave Parameters for Special Cases
  – Lossy dielectrics (general case)
  – Good dielectrics
  – Good conductors
Complex Wave Parameters

The Complex Permittivity, $\tilde{\varepsilon}$

There are two ways to specify the electrical properties of a material:

Complex Permittivity: $\tilde{\varepsilon} = \varepsilon' - j\varepsilon''$  
Real Permittivity & Conductivity: $\varepsilon$ and $\sigma$

We can relate the two systems above using Maxwell's equations.

Complex Permittivity: $\nabla \times \vec{H} = j\omega \varepsilon \vec{E}$

Real Permittivity & Conductivity: $\nabla \times \vec{H} = \vec{J} + j\omega \varepsilon \vec{E}$  
$= \sigma \vec{E} + j\omega \varepsilon \vec{E}$  
$= (\sigma + j\omega \varepsilon) \vec{E}$  
$= j\omega \left( \frac{\sigma}{j\omega} + \varepsilon \right) \vec{E}$

The relation is:

$\tilde{\varepsilon} = \varepsilon' - j\varepsilon'' = \varepsilon + \frac{\sigma}{j\omega}$
### Parameter Values for Various Materials

#### Complex Permittivity

<table>
<thead>
<tr>
<th>Material under test</th>
<th>$\varepsilon'$</th>
<th>$\varepsilon''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air in an empty container</td>
<td>0.937 – 1.010</td>
<td>−0.020 ~ +0.007</td>
</tr>
<tr>
<td>Water collected from a laboratory tap</td>
<td>10.030 – 11.949</td>
<td>16.783 ~ 14.759</td>
</tr>
</tbody>
</table>

#### Conductivity

<table>
<thead>
<tr>
<th>Material</th>
<th>Conductivity $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>$5.96 \times 10^{7}$ S/m</td>
</tr>
<tr>
<td>Gold</td>
<td>$4.10 \times 10^{7}$ S/m</td>
</tr>
<tr>
<td>Nickel</td>
<td>$1.43 \times 10^{7}$ S/m</td>
</tr>
<tr>
<td>Iron</td>
<td>$1.00 \times 10^{7}$ S/m</td>
</tr>
<tr>
<td>Drinking Water</td>
<td>$5 \times 10^{-3}$ S/m</td>
</tr>
<tr>
<td>Air</td>
<td>$\sim 10^{-16}$ S/m</td>
</tr>
<tr>
<td>Teflon</td>
<td>$\sim 10^{-24}$ S/m</td>
</tr>
</tbody>
</table>

### The Complex Permeability, $\tilde{\mu}$

Similarly, the permeability can also be a complex number.

\[ \mu = \mu' - j\mu'' \]

It is unusual to see complex permeability used in practice.
The Complex Wave Number, $k$

A wave travelling in the $+z$ direction can be written in terms of the wave number $k$ as

$$E(z) = \tilde{P}e^{-jkz} \quad k = k' - jk''$$

Substituting this into the wave solution yields

$$E(z) = \tilde{P}e^{-j(k' - jk'')z} = \tilde{P}e^{-k'z}e^{-jk''z}$$

The Complex Propagation Constant, $\gamma$

A wave travelling the $+z$ direction can be written in terms of the complex propagation constant $\gamma$ as

$$E(z) = \tilde{P}e^{-\gamma z} \quad \gamma = \gamma' + j\gamma''$$

Substituting this into the wave solution yields

$$E(z) = \tilde{E}_0e^{-(\gamma' + j\gamma'')z} = \tilde{E}_0e^{-\gamma'z}e^{-j\gamma''z}$$
Attenuation Coefficient $\alpha$ and Phase Constant $\beta$

A wave travelling the $+z$ direction can also be written in terms of an attenuation coefficient $\alpha$ and a phase constant $\beta$ and as

\[
\begin{align*}
\vec{E}(z) &= \vec{E}_0 e^{-k_z z} e^{-jk_z z} \\
\vec{E}(z) &= \vec{E}_0 e^{-\alpha z} e^{-j\beta z} \\
\end{align*}
\]

We now have the physical meaning of the real and imaginary parts of the wave vector $k$ and propagation constant $\gamma$.

\[
\begin{align*}
k &= \beta - j\alpha \\
\alpha &= -\text{Im}[k] \\
\beta &= \text{Re}[k] \\
\gamma &= \alpha + j\beta \\
\alpha &= \text{Re}[\gamma] \\
\beta &= \text{Im}[\gamma] \\
\end{align*}
\]

Physical Meaning of $\alpha$ and $\beta$

The equation of the wave is

\[
E(z) = E_0 e^{-\alpha z} e^{-j\beta z}
\]

$\beta$ takes on the meaning of the wave vector we discussed up to this point.

\[
\beta = \frac{2\pi}{\lambda} = k_0 n
\]
Calculating $\alpha$ and $\beta$ from $\mu$, $\varepsilon$, and $\sigma$

Given complex permeability and permittivity,

$$k = \beta - j\alpha = \omega\sqrt{\mu\varepsilon} \quad \rightarrow \quad \alpha = -\text{Im}\left[\omega\sqrt{\mu\varepsilon}\right]$$

$$\beta = \text{Re}\left[\omega\sqrt{\mu\varepsilon}\right]$$

Given real permeability, permittivity and conductivity,

$$-\gamma^2 = \omega^2\mu\varepsilon = \omega^2\mu(\varepsilon + \sigma/j\omega)$$

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$(\alpha + j\beta)^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$(\alpha^2 - \beta^2) + j(2\alpha\beta) = (-\omega^2\mu\varepsilon) + j(\omega\mu\sigma)$$

$$\downarrow$$

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

$$2\alpha\beta = \omega\mu\sigma$$

$$\alpha = \omega \frac{\mu\varepsilon}{2} \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1$$

$$\beta = \omega \frac{\mu\varepsilon}{2} \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1$$

$\alpha$ collects all loss information into a single parameter.

$\beta$ collects all phase information into a single parameter.

Both are an unintuitive mix of the fundamental parameters.

Absorption Coefficient, $\alpha_p$

The absorption coefficient $\alpha_p$ describes how power decays as a function of position.

$$P(z) = P_0 e^{-\alpha_p z}$$

We previously defined the attenuation coefficient $\alpha$ that described how the field amplitude decays as a function of position.

$$E(z) = E_0 e^{-\alpha z} e^{-j\beta z}$$

Given that $P \propto E^2$, the attenuation coefficient $\alpha$ and absorption coefficient $\alpha_p$ are related through

$$P(z) = \frac{|E(z)|^2}{\eta} = \frac{E_0^2}{\eta} e^{-2\alpha z} \quad \alpha_p = 2\alpha$$
Waves with Complex $k$

- **Purely Real $k$**
  - Uniform amplitude
  - Oscillations move power
  - Considered to be a propagating wave

- **Purely Imaginary $k$**
  - Decaying amplitude
  - No oscillations, no flow of power
  - Considered to be evanescent

- **Complex $k$**
  - Decaying amplitude
  - Oscillations move power
  - Considered to be a propagating wave (not evanescent)

This implies that these are the only 2.5 configurations that electromagnetic fields can take on.

2D Waves with Doubly Complex $k$

- **Real $k_x$**
  - $k_x$ real

- **Imaginary $k_x$**
  - $k_x$ imaginary

- **Complex $k_x$**
  - $k_x$ complex

- **Real $k_y$**
  - $k_y$ real

- **Imaginary $k_y$**
  - $k_y$ imaginary

- **Complex $k_y$**
  - $k_y$ complex

This implies that these are the only 2.5 configurations that electromagnetic fields can take on.
Complex Impedance $\eta$

The wave impedance is in general a complex number.

$$\eta = |\eta| \angle \theta_\eta = R_0 + jX_0$$

The amplitude/phase form is the most meaningful when substituted into the expression for the magnetic field component of a wave.

$$\vec{H} = \frac{k \times \vec{P}}{\eta} e^{-j(k \cdot \vec{r})} = \frac{k \times \vec{P}}{|\eta|} e^{-j(k \cdot \vec{r} + \theta_\eta)}$$

- $|\eta|$ affects magnitude
- $\theta_\eta$ affects phase

Impedance $\eta$ in Terms of $\mu$, $\varepsilon$, and $\sigma$

Given complex permeability and permittivity,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Given real permeability, permittivity and conductivity,

$$\eta = \sqrt{\frac{\mu}{\varepsilon + \sigma/j \omega}} = \sqrt{\frac{\mu/\varepsilon}{1 + \sigma/j \omega}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{1 + (\sigma/j \omega)^2}$$

$$\angle \eta = \frac{1}{2} \tan^{-1} \left( \frac{\sigma}{\omega \varepsilon} \right)$$

$\eta$ collects all amplitude and phase information between $E$ and $H$ into a single parameter.

It is an unintuitive mix of the fundamental parameters.
Complex Refractive Index, $\tilde{n}$ (3 of 3)

Recall that $k = k_0n$. However, we now know that $k$ is a complex number, so refractive index must be as well.

\[ \tilde{n} = n_o - j\kappa \]

Extinction coefficient, \( \kappa \)

Ordinary refractive index, \( n_o \)

We can now relate the real and imaginary parts of refractive index to the real and imaginary parts of \( k \) as well as \( \alpha \) and \( \beta \).

\[
\begin{align*}
    k &= k_0\tilde{n} \\
    k' - jk'' &= k_0(n_o - j\kappa) \\
    \beta - j\alpha &= k_0(n_o - j\kappa)
\end{align*}
\]

\[
\begin{align*}
    n_o &= \text{Re}[k] = \frac{\beta}{k_0} \\
    \kappa &= \text{Im}[k] = \frac{\alpha}{k_0}
\end{align*}
\]

Complex Refractive Index for Various Materials

Gold

Crystalline Silicon (Si)

Copper

Amorphous Silicon

http://www.ioffe.ru/SVA/NSM/nk/
Loss Tangent

Sometimes material loss is given in terms of a “loss tangent.”

\[ \tan \delta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega \varepsilon} \quad P(z) = P_0 e^{-\delta k_0 nz} \]

Recall that interpreting wave properties (velocity and loss) is not intuitive using just the complex dielectric function. To do this, we preferred the complex refractive index.

It turns out that the loss tangent and the extinction coefficient are essentially the same quantity.

\[ \delta = \frac{2\kappa}{n} = \frac{\alpha_{\text{abs}}}{k_0 n} \]

It is called a loss tangent because it is the angle in the complex plane formed between the resistive component and the reactive component of the electromagnetic field.

Visualization of EM Waves
Waves in Vacuum

- $H$ is $377\times$ smaller than $E$.
  \[ \eta_v = \frac{E_v}{H_v} \approx 376.73 \, \Omega \]
- $E$ and $H$ are in phase
  \[ \text{Im}[\eta] = 0 \]
- $E \perp H$
  \[ \vec{H} \propto \vec{k} \times \vec{P} \]
- Amplitude does not decay
  \[ \sigma = 0 \]

Waves in Dielectric

- $H$ is larger now, but still smaller than $E$.
  \[ \eta = \frac{1}{\sqrt{\epsilon}} \]
- $E$ and $H$ are still in phase
  \[ \text{Im}[\eta] = 0 \]
- $E \perp H$
  \[ \vec{H} \propto \vec{k} \times \vec{P} \]
- Amplitude still does not decay
  \[ \sigma = 0 \]
Waves in Lossy Dielectric

- $H$ remains larger, but still smaller than $E$.
- $E$ and $H$ are out of phase!
- $\text{Im}[\eta] \neq 0$
- $E \perp H$
- $\tilde{H} \propto \hat{k} \times \tilde{P}$
- Amplitude decays
  $\sigma \neq 0$

More Realistic Wave ($E$ Only)

It is important to remember that plane waves are of infinite extent in the $x$ and $y$ directions.
More Realistic Wave ($E$ & $H$)

It is important to remember that plane waves are of infinite extent in the $x$ and $y$ directions.

Complex Wave Parameters for Special Cases
Summary of Waves in Lossy Dielectrics

Condition: This is the general case. All materials have loss.

Fundamental Parameters: \( \sigma, \mu = \mu_0 \mu_i, \quad \vec{E} = \epsilon_0 \vec{E}_i = \epsilon + \frac{\sigma}{j \omega} \)

Attenuation Coefficient: \( \alpha = -\text{Im}[k] \quad \alpha = \omega \sqrt{\frac{\mu \epsilon}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right] \)

Phase Constant: \( \beta = \text{Re}[k] \quad \beta = \omega \sqrt{\frac{\mu \epsilon}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \epsilon} \right)^2} + 1 \right] \)

Impedance: \( \eta = \frac{\beta}{\epsilon} = \frac{\mu \sqrt{\epsilon}}{1 + \sigma / j \omega \epsilon} \quad |\eta| = \frac{\sqrt{\mu / \epsilon}}{\left[ 1 + \left( \sigma / j \omega \epsilon \right)^2 \right]^{1/4}} \quad \angle \eta = \frac{1}{2} \tan^{-1} \left( \frac{\sigma}{\omega \epsilon} \right) \)

Summary of Waves in Lossless Dielectrics

Condition: \( \sigma \ll \omega \epsilon \)

Fundamental Parameters: \( \sigma \approx 0, \quad \mu = \mu_0 \mu_i, \quad \epsilon = \epsilon_0 \epsilon_i \)

Attenuation Coefficient: \( \alpha = 0 \quad \text{No attenuation} \)

Phase Constant: \( \beta = \omega \sqrt{\mu \epsilon} \quad \text{H is } \sim 3\text{x} \text{ small than } E. \quad E \text{ and } H \text{ are in phase} \)

Impedance: \( \eta = \frac{\mu}{\epsilon} = \eta_0 \frac{\mu}{\epsilon_i} \quad |\eta| = \sqrt{\mu / \epsilon} \quad \angle \eta = 0 \)

Notes:
- Most commonly analyzed, due to easy math.
- Usually a good approximation for dielectrics.
- Not physically real, except in vacuum. All materials have loss.
Summary of Waves in Good Conductors

**Condition:** \( \sigma \gg \omega \varepsilon \)

**Fundamental Parameters:** \( \sigma, \mu = \mu_0 \mu_i, \varepsilon = \varepsilon_0 \varepsilon_i \)

**Attenuation Coefficient:**

\[
\alpha = \beta = \sqrt{\frac{\omega \mu \sigma}{2}} \quad \text{Strong attenuation}
\]

**Phase Constant:**

\[
\alpha = \beta = \sqrt{\frac{\omega \mu \sigma}{2}}
\]

\( E \) and \( H \) are out of phase.

**Impedance:**

\[
\eta = \sqrt{\frac{\omega \mu}{\sigma}} \quad |\eta| = \sqrt{\frac{\omega \mu}{\sigma}} \quad \angle \eta = 45^\circ
\]

**Notes:**

- Very strong attenuation.
- Waves tend to reflect from good conductors so often do not experience the loss.
- \( E \) leads \( H \) by 45°.