EE 4347
Applied Electromagnetics

Topic 3g

Scattering at an Interface: Oblique Incidence

Lecture Outline

• Geometry of a Plane Wave at Oblique Incidence
• Boundary condition for $\hat{k}$
• Angle of reflection & refraction
• Fresnel equations
• Reflectance and Transmittance
• Example – Plot of Reflectance and Transmittance
Geometry of a Plane Wave at Oblique Incidence

We start with a perfectly flat interface between two materials. For mathematical convenience, we let the interface lie exactly in the $xy$ plane. We must draw our coordinates so it is a right-handed system.

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$
Let there be a wave described by the wave vector $\vec{k}_{inc}$ be incident onto the interface from above.

Recall the wave vector is:

$$\vec{k}_{inc} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$$

$$|\vec{k}_{inc}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$= \frac{2\pi}{\lambda}$$

$$= \omega \sqrt{\mu \varepsilon}$$

The wave vector $\vec{k}_{inc}$ and the surface normal $\hat{a}$ define a plane. Both of these vectors lie within this plane.

This is called the *plane of incidence*. 
The wave vector is defined by two angles, $\theta$ and $\phi$.

$\theta$ = elevation angle
$\phi$ = azimuthal angle

The components of the incident wave vector can be calculated according to

$$\vec{k}_{\text{inc}} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$$

$$k_x = k_0 n_i \cos \phi \sin \theta$$
$$k_y = k_0 n_i \sin \phi \sin \theta$$
$$k_z = k_0 n_i \cos \theta$$

The choice of directions for polarization becomes important when a device is involved.

The TE polarization is defined to be perpendicular to the plane of incidence.

$$\hat{a}_{\text{TE}} = \frac{\hat{a}_z \times \vec{k}_{\text{inc}}}{|\hat{a}_z \times \vec{k}_{\text{inc}}|}$$

TE polarization
s polarization
$\perp$ polarization
The TM polarization is defined to be parallel to the plane of incidence.

\[ \hat{a}_{TM} = \frac{\hat{a}_{TE} \times \hat{k}_{inc}}{|\hat{a}_{TE} \times \hat{k}_{inc}|} \]

TM polarization
p polarization
|| polarization

The incident wave, reflected wave, and transmitted wave all lie within the plane of incidence.
Since everything happens within the plane of incidence, we can rotate the plane of incidence to something more convenient to analyze.

Let the plane of incidence lie in the \(xz\) plane.

\[
\hat{a}_{\text{TE}} \rightarrow \hat{a}_y, \quad k_y \rightarrow 0
\]

This rotation is valid for calculating angle of reflection, angle of refraction, and amplitude of the reflected and transmitted waves. However, vector quantities like \(\hat{k}_\text{inc}, \hat{a}_{\text{TE}}, \hat{a}_{\text{TM}},\) and \(\vec{P}\) will be different in the rotated system.

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**Example #1 – TE and TM Directions**

In spherical coordinates, a wave is incident at \(\theta = 30^\circ\) and \(\phi = 120^\circ\) from air into water. Calculate unit vectors in the TE and TM directions.

**Solution**

First, the incident wave vector is needed. Since only direction is of concern, the magnitude \(k_0n_1\) can be ignored.

\[
\hat{k}_x = k_0n_1 \cos \phi \sin \theta = \cos(120^\circ) \sin(30^\circ) = -0.25
\]
\[
\hat{k}_z = k_0n_1 \sin \phi \sin \theta = \sin(120^\circ) \sin(30^\circ) = 0.4330
\]
\[
\hat{k}_z = k_0n_1 \cos \theta = \cos(30^\circ) = 0.8660
\]

The TE direction is

\[
\hat{a}_{\text{TE}} = \frac{\hat{a}_z \times \hat{k}_\text{inc}}{|\hat{a}_z \times \hat{k}_\text{inc}|} = \frac{(0, 0, 1) \times (-0.25, 0.4330, 0.8660)}{|(0, 0, 1) \times (-0.25, 0.4330, 0.8660)|} = (-0.8660, -0.5, 0)
\]
Example #1 – TE and TM Directions

In spherical coordinates, a wave is incident at $\theta = 30^\circ$ and $\phi = 120^\circ$ from air into water. Calculate unit vectors in the TE and TM directions.

**Solution cont’d**

The TM direction is

$$\hat{a}_{TM} = \frac{\hat{a}_{TE} \times \vec{k}_{inc}}{|\hat{a}_{TE} \times \vec{k}_{inc}|}$$

$$= \frac{(-0.8660, -0.5, 0) \times (-0.25, 0.4330, 0.8660)}{|(-0.8660, -0.5, 0) \times (-0.25, 0.4330, 0.8660)|}$$

$$= (-0.4330, 0.75, -0.5)$$

Boundary Condition for $\vec{k}$
Without making any assumptions, our waves can be written as

\[ E_x(\vec{r}) = \hat{E}_x e^{-j\beta_x x} = \hat{E}_{x,0} e^{-j\beta_{x,0} x} e^{-j\beta_{x,0} z} \]

\[ E_y(\vec{r}) = \hat{E}_y e^{-j\beta_y y} = \hat{E}_{y,0} e^{-j\beta_{y,0} y} e^{-j\beta_{y,0} z} \]

\[ E_z(\vec{r}) = \hat{E}_z e^{-j\beta_z z} = \hat{E}_{z,0} e^{-j\beta_{z,0} z} \]

Remember \( k_z = 0 \) for our analysis.

However, we only care about what is happening exactly on the interface so we set \( z = 0 \).

\[ E_x(z = 0) = \hat{E}_{x,0} e^{-j\beta_{x,0} x} e^0 = \hat{E}_{x,0} e^{-j\beta_{x,0} x} \]

\[ E_y(z = 0) = \hat{E}_{y,0} e^{-j\beta_{y,0} y} e^0 = \hat{E}_{y,0} e^{-j\beta_{y,0} y} \]

\[ E_z(z = 0) = \hat{E}_{z,0} e^{-j\beta_{z,0} z} e^0 = \hat{E}_{z,0} e^{-j\beta_{z,0} z} \]

Boundary conditions require the tangential components of \( E \) to be continuous across the interface.

\[ E_{x,0} = E_{x,0} \]

\[ E_{x,0} + E_{x,1} = E_{x,1} \]

\[ E_{x,0} e^{-j\beta_{x,0} x} + E_{x,1} e^{-j\beta_{x,1} x} = E_{x,1} e^{-j\beta_{x,1} x} \]
Boundary Condition for $\vec{k}$ (3 of 3)

The only possible way these equations can be satisfied is if

$$k_{x,i} = k_{x,r} = k_{x,t}$$

$k_x$ is the tangential component of the wave vector. We generalize this to any orientation of the plane of incidence as

$$\vec{k}_{1,tan} = \vec{k}_{2,tan}$$

The tangential components of $\vec{k}$ are continuous across the interface.

We can also conclude from this that the incident wave, reflected wave, and transmitted wave all lie within the plane of incidence.

What About $k_{z,i}$ and $k_{z,t}$?

The vector components of a plane wave must satisfy the dispersion relation of the medium the wave is in.

The dispersion relations for the incident, reflected, and transmitted waves are

$$|\vec{k}|^2 = (k_0n_1)^2 = k_{x,i}^2 + k_{z,i}^2$$

$$|\vec{k}_r|^2 = (k_0n_1)^2 = k_{x,t}^2 + k_{z,t}^2$$

$$|\vec{k}|^2 = (k_0n_2)^2 = k_{x,i}^2 + k_{z,t}^2$$

However we know that $k_{x,i} = k_{x,r} = k_{x,t}$ so we just call all of these $k_x$.

$$|\vec{k}|^2 = (k_0n_1)^2 = k_x^2 + k_{z,i}^2 \quad \rightarrow \quad k_{x,z}^2 = (k_0n_1)^2 - k_x^2$$

$$|\vec{k}_r|^2 = (k_0n_1)^2 = k_x^2 + k_{z,t}^2 \quad \rightarrow \quad k_{x,z}^2 = (k_0n_1)^2 - k_x^2$$

$$|\vec{k}|^2 = (k_0n_2)^2 = k_x^2 + k_{z,t}^2 \quad \rightarrow \quad k_{x,z}^2 = (k_0n_2)^2 - k_x^2$$
Reflected Wave, $k_{z,r}$

From the previous slide, the dispersion relation for the incident and reflected waves were

$$k_{z,i}^2 = (k_0n_1)^2 - k_z^2$$
$$k_{z,r}^2 = (k_0n_1)^2 - k_z^2$$

From these, we see that $k_{z,r}^2 = k_{z,i}^2$.

Therefore $k_{z,r} = \pm k_{z,i}$.

We resolve the sign by recognizing that the reflected wave must be propagating in the $-z$ direction.

We conclude that

$$k_{z,r} = -k_{z,i}$$

Example #2 – Transmitted Wave Vector

In spherical coordinates, a wave is incident at $\theta = 30^\circ$ and $\phi = 120^\circ$ from air into water. Calculate the wave vector of the transmitted wave if the wavelength $\lambda_2$ in water is 6.2832 m.

Solution

The incident wave vector is

$$k_{ix} = k_0n_1 \cos \phi \sin \theta = k \cos \phi \sin \theta = (2\pi/\lambda) \cos \phi \sin \theta$$
$$k_{iy} = k_0n_1 \sin \phi \sin \theta = k \sin \phi \sin \theta = (2\pi/\lambda) \sin \phi \sin \theta$$
$$k_{iz} = k_0n_1 \cos \theta = k \cos \theta = (2\pi/\lambda) \cos \theta$$

$$= (2\pi/6.2832) \cos(120^\circ) \sin(30^\circ) = -0.25 \text{ m}^{-1}$$
$$= (2\pi/6.2832) \sin(120^\circ) \sin(30^\circ) = 0.4330 \text{ m}^{-1}$$
$$= (2\pi/6.2832) \cos(30^\circ) = 0.8660 \text{ m}^{-1}$$
Example #2 – Transmitted Wave Vector

In spherical coordinates, a wave is incident at $\theta = 30^\circ$ and $\phi = 120^\circ$ from air into water. Calculate the wave vector of the transmitted wave if the wavelength $\lambda_2$ in water is 6.2832 m.

Solution cont’d

Tangential components of the wave vector are continuous across the interface.

\[
\begin{align*}
  k_{1x} &= -0.25 \quad \rightarrow k_{2x} = k_{1x} = -0.25 \text{ m}^{-1} \\
  k_{1y} &= 0.4330 \quad \rightarrow k_{2y} = k_{1y} = 0.4330 \text{ m}^{-1} \\
  k_{1z} &= 0.8660
\end{align*}
\]

The longitudinal component of the wave vector is calculated from the dispersion relation.

\[
\begin{align*}
  k_{2x}^2 + k_{2y}^2 + k_{2z}^2 &= |k_2|^2 \\
  \rightarrow k_{2z} &= \sqrt{|k_2|^2 - k_{2x}^2 - k_{2y}^2} = \sqrt{\left(\frac{2\pi}{\lambda_2}\right)^2 - k_{2x}^2 - k_{2y}^2}
\end{align*}
\]

Altogether, the transmitted wave vector is

\[
\vec{k}_t = -0.25 \hat{a}_x + 0.4330 \hat{a}_y + 1.2324 \hat{a}_z \text{ m}^{-1}
\]
Law of Reflection

Let’s relate the angle of the incident and reflected wave based on what we know about the wave vector components.

\[ k_x = k_0 n_1 \sin \theta \]
\[ k_y = 0 \]
\[ k_{z,i} = k_0 n_1 \cos \theta \]

\[ \theta_r = \theta_i \]

\[ k_{z,r} = -k_{z,i} \]
### Geometry of Reflection and Refraction

Snell’s Law

Recall the dispersion relations for the incident and transmitted waves.

\[(k_0 n_1)^2 = k_z^2 + k_{x,i}^2 \quad \text{and} \quad (k_0 n_2)^2 = k_z^2 + k_{x,t}^2\]

Solving both of these equations for \(k_z^2\) gives

\[k_z^2 = (k_0 n_1)^2 - k_{x,i}^2 \quad \text{and} \quad k_z^2 = (k_0 n_2)^2 - k_{x,t}^2\]

The right-hand side of these equations must be equal.

\[(k_0 n_1)^2 - k_{x,i}^2 = (k_0 n_2)^2 - k_{x,t}^2\]

\[(k_0 n_1 \cos \theta_i)^2 = (k_0 n_2 \cos \theta_i)^2\]

\[(k_0 n_1)^2 - (k_0 n_1 \cos \theta_i)^2 = (k_0 n_2)^2 - (k_0 n_2 \cos \theta_i)^2\]

\[n_1^2 (1 - \cos^2 \theta_i) = n_2^2 (1 - \cos^2 \theta_i)\]

\[n_1^2 \sin^2 \theta_i = n_2^2 \sin^2 \theta_t \quad \rightarrow \quad n_1 \sin \theta_i = n_2 \sin \theta_t\]
Summary of Scattering Angles

Snell’s Law
\[ n_1 \sin \theta_i = n_2 \sin \theta_i \]

Law of Reflection
\[ \theta_r = \theta_i \]

Animation of Reflection & Refraction

Law of Reflection
\[ \theta_i = \theta_r \]

Snell’s Law
\[ n_1 \sin \theta_i = n_2 \sin \theta_i \]
Example #3 – Scattering Angles

In spherical coordinates, a wave is incident at $\theta = 30^\circ$ and $\phi = 120^\circ$ from air into water. Determine the angle of incidence $\theta_i$, angle of reflection $\theta_r$, and angle of transmission $\theta_t$.

Solution

The angle of incidence $\theta_i$ was given in the problem to be $\theta_i = 30^\circ$.

From the law of reflection, the angle of reflection $\theta_r$ is $\theta_r = \theta_i = 30^\circ$.

From Snell’s law, the angle of transmission $\theta_t$ is

$$\theta_t = \sin^{-1}\left(\frac{n_1 \sin \theta_i}{n_2}\right) = \sin^{-1}\left(\frac{1.0}{1.33} \sin 30^\circ\right) = 22^\circ$$

The Fresnel Equations
Fresnel Equations

TE, s, \perp\ Polarization

\[ r_{TE} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \]
\[ t_{TE} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \]
\[ 1 + r_{TE} = t_{TE} \]

TM, p, \parallel\ Polarization

\[ r_{TM} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \]
\[ t_{TM} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} \]
\[ 1 + r_{TM} = \frac{\cos \theta_i}{\cos \theta_t} t_{TM} \]

Law of reflection and Snell’s law tells us the directions of the reflected and transmitted waves relative to the incident wave.

The Fresnel equations tell how much gets reflected and transmitted.

These equations are valid for lossy media as long as the impedances and angles are allowed to be complex numbers.

Derivation of TE Fresnel Equations (1 of 5)

Start with the general expressions for the incident, reflected, and transmitted waves for the TE polarization.

\[ \vec{E}_i = (E_{0,0}\hat{\imath}) e^{-j\beta_0 x \cos \theta} \]
\[ \vec{E}_r = (E_{0,0}\hat{\imath}) e^{-j\beta_0 x \cos \theta} \]
\[ \vec{E}_t = (E_{0,0}\hat{\imath}) e^{-j\beta_0 x \cos \theta} \]

\[ \vec{H}_i = \frac{E_{0,0}}{\eta_0} (-\cos \theta \hat{\imath} + \sin \theta \hat{\jmath}) e^{-j\beta_0 x \cos \theta} \]
\[ \vec{H}_r = \frac{E_{0,0}}{\eta_0} (\cos \theta \hat{\imath} + \sin \theta \hat{\jmath}) e^{-j\beta_0 x \cos \theta} \]
\[ \vec{H}_t = \frac{E_{0,0}}{\eta_2} (-\cos \theta \hat{\imath} + \sin \theta \hat{\jmath}) e^{-j\beta_0 x \cos \theta} \]
The boundary conditions at the interface are

\[
\dot{E}_i(0) + \dot{E}_e(0) = \dot{E}_i(0).
\]

\[
E_{0,i}e^{-j\beta_{ni}(\sin\theta_i)} + E_{0,e}e^{-j\beta_{ne}(\sin\theta_e)} = E_{0,i}e^{-j\beta_{ni}(\sin\theta_i)}.
\]

\[
E_{0,i}e^{-j\beta_{ni}(\sin\theta_i)} + E_{0,e}e^{-j\beta_{ne}(\sin\theta_e)} = E_{0,i}e^{-j\beta_{ni}(\sin\theta_i)}.
\]

\[
E_{0,i} + E_{0,e} = E_{0,i}.
\]

**Snell’s Law**

\[
\frac{E_{0,i}}{\eta_i}(-\cos\theta_i)e^{-j\beta_{ni}(\sin\theta_i)} + \frac{E_{0,e}}{\eta_e}(\cos\theta_i)e^{j\beta_{ne}(\sin\theta_e)} = \frac{E_{0,i}}{\eta_i}(-\cos\theta_i)e^{-j\beta_{ni}(\sin\theta_i)}.
\]

\[
\frac{E_{0,i}}{\eta_i}(-\cos\theta_i)e^{-j\beta_{ni}(\sin\theta_i)} + \frac{E_{0,e}}{\eta_e}(\cos\theta_i)e^{-j\beta_{ne}(\sin\theta_e)} = \frac{E_{0,i}}{\eta_i}(-\cos\theta_i)e^{-j\beta_{ni}(\sin\theta_i)}.
\]

\[
\frac{E_{0,i}}{\eta_i}\cos\theta_i = \frac{E_{0,e}}{\eta_e}\cos\theta_i
\]

\[
\frac{E_{0,i}}{\eta_i}\cos\theta_i = \frac{E_{0,e}}{\eta_e}\cos\theta_i
\]

\[
E_{0,i} + E_{0,e} = E_{0,i}
\]

Eq. (1a) \hspace{1cm} \frac{E_{0,i}}{\eta_i}\cos\theta_i = \frac{E_{0,e}}{\eta_e}\cos\theta_i \hspace{1cm} \text{Eq. (1b)}

Substituting Eq. (1a) into Eq. (1b) to eliminate \(E_{0,i}\) gives

\[
\frac{E_{0,i}\cos\theta_i}{\eta_i} = \frac{E_{0,e}\cos\theta_i}{\eta_e}
\]

\[
\frac{E_{0,i}\cos\theta_i}{\eta_i} = \frac{E_{0,e}\cos\theta_i}{\eta_e}
\]

\[
\frac{E_{0,i}}{E_{0,i}} = \frac{\eta_i\cos\theta_i}{\eta_e}\frac{\cos\theta_i}{\eta_i} + \frac{\cos\theta_i}{\eta_e}
\]

\[
R_{TE} = \frac{E_{0,e}}{E_{0,i}} = \frac{\eta_e\cos\theta_i}{\eta_i\cos\theta_i + \eta_e\cos\theta_i}
\]
Derivation of TE Fresnel Equations (4 of 5)

\[ E_{0,i} + E_{0,r} = E_{0,i} \quad \text{Eq. (1a)} \]

Substituting the expression for \( r \) into \( \text{Eq. (1a)} \) to eliminate \( E_{0,r} \) gives

\[ E_{0,i} + E_{0,r} = E_{0,i} \]
\[ E_{0,i} + E_{0,i} \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} = E_{0,i} \]
\[ E_{0,i} \left( 1 + \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \right) = E_{0,i} \]
\[ E_{0,i} \left( 1 + \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \right) = E_{0,i} \]
\[ E_{0,i} \frac{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i + \eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \]
\[ E_{0,i} \frac{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \]

\[ r_{TE} = \frac{E_{0,x}}{E_{0,i}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \]

\[ t_{TE} = \frac{E_{0,t}}{E_{0,i}} = \frac{2 \eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i} \]

Derivation of TE Fresnel Equations (5 of 5)

\[ E_{0,i} + E_{0,r} = E_{0,i} \quad \text{Eq. (1a)} \]

Divide \( \text{Eq. (1a)} \) by \( E_{0,i} \) to get

\[ E_{0,i} + E_{0,r} = E_{0,i} \]
\[ E_{0,i} + E_{0,i} = E_{0,i} \]
\[ \frac{E_{0,i} + E_{0,r}}{E_{0,i}} = \frac{E_{0,i}}{E_{0,i}} \]
\[ \frac{E_{0,i} + E_{0,r}}{E_{0,i}} = \frac{E_{0,i}}{E_{0,i}} \]

\[ 1 + r_{TE} = t_{TE} \]
Derivation of TM Fresnel Equations (1 of 6)

Start with the general expressions for the incident, reflected, and transmitted waves for the TM polarization.

\[ E_i = E_{0,i} \left( \cos \theta_i \hat{a}_x - \sin \theta_i \hat{a}_z \right) e^{-j\beta_{0,i} (\sin \theta_i \chi + \cos \theta_i \zeta)} \]

\[ E_r = E_{0,r} \left( \cos \theta_i \hat{a}_x + \sin \theta_i \hat{a}_z \right) e^{-j\beta_{0,r} \theta_i} \]

\[ E_t = E_{0,t} \left( \cos \theta_i \hat{a}_x - \sin \theta_i \hat{a}_z \right) e^{-j\beta_{0,t} (\sin \theta_i \chi - \cos \theta_i \zeta)} \]

\[ H_i = \left( \frac{E_{0,i}}{\eta_i} \hat{a}_y \right) e^{-j\beta_{0,i} (\sin \theta_i \chi + \cos \theta_i \zeta)} \]

\[ H_r = \left( -\frac{E_{0,r}}{\eta_i} \hat{a}_y \right) e^{-j\beta_{0,r} (\sin \theta_i \chi - \cos \theta_i \zeta)} \]

\[ H_t = \left( \frac{E_{0,t}}{\eta_2} \hat{a}_y \right) e^{-j\beta_{0,t} (\sin \theta_i \chi - \cos \theta_i \zeta)} \]

Derivation of TM Fresnel Equations (2 of 6)

The boundary conditions for the electric field at the interface are

\[ E_i(0) \left|_{\chi=0} + E_t(0) \left|_{\chi=0} = E_r(0) \right|_{\chi=0} \]

\[ E_{0,i} \left( \cos \theta_i \hat{a}_x \right) e^{-j\beta_{0,i} (\sin \theta_i \chi)} + E_{0,t} \left( \cos \theta_i \hat{a}_x \right) e^{-j\beta_{0,t} (\sin \theta_i \chi)} = E_{0,r} \left( \cos \theta_i \hat{a}_x \right) e^{-j\beta_{0,r} \theta_i} + E_{0,t} \left( \cos \theta_i \hat{a}_x \right) e^{-j\beta_{0,t} (\sin \theta_i \chi)} \]

\[ E_{0,i} \cos \theta_i + E_{0,r} \cos \theta_i = E_{0,t} \cos \theta_i + E_{0,t} \cos \theta_i \]

\[ E_{0,i} \cos \theta_i + E_{0,r} \cos \theta_i = E_{0,t} \cos \theta_i + E_{0,t} \cos \theta_i \]

\[ E_{0,i} + E_{0,r} = E_{0,t} \frac{\cos \theta_i}{\cos \theta_i} \]
Derivation of TM Fresnel Equations (3 of 6)

The boundary conditions for the magnetic field at the interface are

\[
\left. \vec{H}_i \right|_{y'} + \left. \vec{H}_r \right|_{y'} = \left. \vec{H}_o \right|_{y'}
\]

\[
\left( \frac{E_{0,i}}{\eta_i} \hat{a}_x \right) e^{-j\delta_0 \sin \theta_i} + \left( \frac{E_{0,r}}{\eta_i} \hat{a}_y \right) e^{-j\delta_0 \sin \theta_i} = \left( \frac{E_{0,o\perp}}{\eta_2} \hat{a}_x \right) e^{-j\delta_0 \sin \theta_o}
\]

\[
\left( \frac{E_{0,i}}{\eta_i} \hat{a}_y \right) e^{-j\delta_0 \sin \theta_i} + \left( \frac{E_{0,r}}{\eta_i} \hat{a}_y \right) e^{-j\delta_0 \sin \theta_i} = \left( \frac{E_{0,o\parallel}}{\eta_2} \hat{a}_y \right) e^{-j\delta_0 \sin \theta_o}
\]

\[
\frac{E_{0,i}}{\eta_i} - \frac{E_{0,r}}{\eta_i} = \frac{E_{0,o}}{\eta_i}
\]

Derivation of TM Fresnel Equations (4 of 6)

\[
E_{0,i} + E_{0,r} = E_{0,o \perp} \frac{\cos \theta_i}{\eta_i} \quad \text{Eq. (1a)}
\]

\[
\frac{E_{0,i}}{\eta_i} - \frac{E_{0,r}}{\eta_i} = \frac{E_{0,o \parallel}}{\eta_i} \quad \text{Eq. (1b)}
\]

Substituting Eq. (1a) into Eq. (1b) to eliminate \( E_{0,t} \) gives

\[
E_{0,i} - E_{0,t} = E_{0,o \perp} \frac{\cos \theta_i}{\eta_i} \quad \text{Eq. (1a)}
\]

\[
\frac{E_{0,i}}{\eta_i} - \frac{E_{0,t}}{\eta_i} = \frac{E_{0,o \parallel}}{\eta_i} \quad \text{Eq. (1b)}
\]

\[
\frac{E_{0,i}}{\eta_i} - \frac{E_{0,t}}{\eta_i} = \frac{E_{0,o \perp}}{\eta_i} \frac{\cos \theta_i}{\eta_i} \quad \text{Eq. (1b)}
\]

\[
E_{0,i} = \frac{1}{\eta_i} \frac{1}{\cos \theta_i} \frac{1}{\eta_i} \frac{1}{\cos \theta_i}
\]

\[
E_{0,t} = \frac{1}{\eta_i} \frac{1}{\cos \theta_i} \frac{1}{\eta_i} \frac{1}{\cos \theta_i}
\]

\[
R_{TM} = \frac{E_{0,t}}{E_{0,i}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_i}
\]
Derivation of TM Fresnel Equations (5 of 6)

\[ E_{0,i} + E_{0,r} = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \text{ Eq. (1a)} \]

Substituting the expression for \( r \) into Eq. (1a) to eliminate \( E_{0,r} \) gives

\[ \frac{E_{0,i}}{E_{0,i}} + E_{0,i} = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \]

\[ \frac{E_{0,i}}{E_{0,i}} + E_{0,i} = E_{0,i} \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta'_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta'_i} = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \]

\[ E_{0,i} \left( 1 + \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta'_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta'_i} \right) = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \]

\[ \frac{E_{0,i}}{E_{0,i}} = \frac{1 + \eta_2 \cos \theta_i - \eta_1 \cos \theta'_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta'_i} \]

\[ r_{TM} = \frac{E_{0,r}}{E_{0,i}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta'_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta'_i} \]

Derivation of TM Fresnel Equations (6 of 6)

\[ E_{0,i} + E_{0,r} = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \text{ Eq. (1a)} \]

Divide Eq. (1a) by \( E_{0,i} \) to get

\[ \frac{E_{0,i}}{E_{0,i}} + E_{0,i} = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \]

\[ \frac{E_{0,i}}{E_{0,i}} + E_{0,i} = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \]

\[ \frac{E_{0,i}}{E_{0,i}} + \frac{E_{0,i}}{E_{0,i}} = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \]

\[ \frac{E_{0,i}}{E_{0,i}} + \frac{E_{0,i}}{E_{0,i}} = E_{0,i} \frac{\cos \theta_i}{\cos \theta'_i} \]

\[ 1 + r_{TM} = \frac{\cos \theta_i}{\cos \theta'_i} \]

\[ t_{TM} = \frac{E_{0,i}}{E_{0,i}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta'_i} \]
Example #4 – Fresnel Equations

In spherical coordinates, a wave is incident at $\theta = 30^\circ$ and $\phi = 120^\circ$ from air into water. Calculate the reflection and transmission coefficients for both TE and TM polarizations.

**Solution**

The material impedances are

$$\eta_1 = \frac{\eta_0}{n_1} = \frac{376.73 \, \Omega}{1.0} = 376.73 \, \Omega \quad \eta_2 = \frac{\eta_0}{n_2} = \frac{376.73 \, \Omega}{1.33} = 283.26 \, \Omega$$

The scattering angles were previously found to be

$$\theta_i = \theta_r = 30^\circ \quad \theta_t = 22^\circ$$

The reflection coefficient for the TE polarization is then

$$r_{TE} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t} = \frac{(283.26 \, \Omega) \cos 30^\circ - (376.73 \, \Omega) \cos 22^\circ}{(283.26 \, \Omega) \cos 30^\circ + (376.73 \, \Omega) \cos 22^\circ} = -0.1749$$

The transmission coefficient for the TE polarization is

$$t_{TE} = 1 + r_{TE} = 1 - 0.1749 = 0.8251$$

The transmission coefficient for the TM polarization is

$$1 + r_{TM} = \frac{\cos \theta_i}{\cos \theta_t} \quad t_{TM} = \frac{\cos \theta_i}{\cos \theta_t} (1 + r_{TM}) = \frac{\cos 30^\circ}{\cos 22^\circ} [1 + (-0.1080)] = 0.8332$$
Reflectance & Transmittance

Fresnel Equations for Lossy Media

The Fresnel equations for lossy media remain the same as lossless media as long as the impedance and angles are made complex.

**TE, s, ⊥ Polarization**

\[
\tilde{r}_{\text{TE}} = \frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} = \tilde{n}_i \cos \tilde{\theta}_i - \tilde{n}_l \cos \tilde{\theta}_l
\]

\[
\tilde{t}_{\text{TE}} = \frac{\tilde{E}_{0,i}}{\tilde{E}_{0,i}} = \frac{2\tilde{n}_l \cos \tilde{\theta}_l}{\tilde{n}_l \cos \tilde{\theta}_l + \tilde{n}_l \cos \tilde{\theta}_l}
\]

\[
1 + \tilde{r}_{\text{TE}} = \tilde{t}_{\text{TE}}
\]

**TM, p, || Polarization**

\[
\tilde{r}_{\text{TM}} = \frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} = \frac{\tilde{n}_l \cos \tilde{\theta}_l - \tilde{n}_l \cos \tilde{\theta}_l}{\tilde{n}_l \cos \tilde{\theta}_l + \tilde{n}_l \cos \tilde{\theta}_l}
\]

\[
\tilde{t}_{\text{TM}} = \frac{\tilde{E}_{0,i}}{\tilde{E}_{0,i}} = \frac{2\tilde{n}_l \cos \tilde{\theta}_l}{\tilde{n}_l \cos \tilde{\theta}_l + \tilde{n}_l \cos \tilde{\theta}_l}
\]

\[
1 + \tilde{r}_{\text{TM}} = \frac{\cos \tilde{\theta}_l}{\cos \tilde{\theta}_l} \tilde{t}_{\text{TM}}
\]
**Definition of $R$ and $T$**

The reflection and transmission coefficients relate the amplitude’s of the reflected and transmitted waves relative to the incident wave.

$$\tilde{r} = \frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} \quad \tilde{t} = \frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}}$$

The reflectance and transmittance describes the fraction of power that is reflected or transmitted from the interface.

$$R = \frac{P_r}{P_i} \quad T = \frac{P_t}{P_i}$$

---

**RMS Power Flow**

In the frequency-domain, the Poynting vector describes power flow. It is calculated from the electric and magnetic fields as

$$\tilde{S}_{\text{avg}} = \tilde{E} \times \tilde{H}^*$$

Recall that the electric and magnetic fields can be expressed as

$$\tilde{E}(\tilde{r}, \omega) = \tilde{E}_0 e^{-jk\cdot r} \quad \tilde{H}(\tilde{r}, \omega) = \tilde{k} \times \frac{\tilde{E}_0}{\omega \mu} e^{-jk\cdot r}$$

Substituting these into the definition of RMS Poynting vector gives

$$\tilde{S} = \left(\tilde{E}_0 e^{-jk\cdot r}\right) \times \left(\frac{\tilde{k} \times \tilde{E}_0}{\omega \mu} e^{-jk\cdot r}\right) = \frac{1}{\omega \mu} \tilde{E}_0 \times \left(\tilde{k} \times \tilde{E}_0\right) e^{-jk\cdot r}$$

$$= \frac{\left|\tilde{E}_0\right|^2}{\omega \mu} \left[\tilde{k} - \hat{P} \left(\tilde{k} \cdot \hat{P}\right)\right] e^{-j2\ln|\tilde{k}|\cdot r}$$
**What Carries Power To and From the Interface?**

The flow of power is described by the Poynting vector $\mathbf{\Phi}$.

However, it is only the components of the Poynting vector that are perpendicular to the interface that carry power to and from the interface.

Power conservation at the interface requires that

$$\mathbf{\Phi}_{z,1} = \mathbf{\Phi}_{z,2}$$

To use this, we need expressions for the total Poynting vector in medium 1 and medium 2.
Poynting Vector in Medium 2

The only wave in medium 2 is the transmitted wave.

\[ \vec{\phi}_2 = \vec{E}_i \times \vec{H}_t' = \frac{1}{\omega \mu_i^*} \vec{E}_i \times \left( \vec{k}_t^* \times \vec{E}_t^* \right) \]

This last form of the equation makes it easy to extract the z-component in a later step.

\[ \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{B} \cdot \vec{A}) \]

Poynting Vectors in Medium 1

The total field in medium 1 is the incident wave plus the reflected wave.

\[ \vec{k}_i = \left( \vec{E}_i + \vec{E}_r \right) \times \left( \vec{H}_i + \vec{H}_r \right) \]

\[ = \left( \vec{E}_i \times \vec{H}_i^* \right) + \left( \vec{E}_i \times \vec{H}_r^* \right) + \left( \vec{E}_r \times \vec{H}_i^* \right) + \left( \vec{E}_r \times \vec{H}_r^* \right) \]

\[ = \frac{1}{\omega \mu_i^*} \left[ \vec{E}_i \times \left( \vec{k}_i^* \times \vec{E}_i^* \right) \right] + \frac{1}{\omega \mu_i^*} \left[ \vec{E}_r \times \left( \vec{k}_i^* \times \vec{E}_r^* \right) \right] + \frac{1}{\omega \mu_i^*} \left[ \vec{E}_i \times \left( \vec{k}_r^* \times \vec{E}_i^* \right) \right] + \frac{1}{\omega \mu_i^*} \left[ \vec{E}_r \times \left( \vec{k}_r^* \times \vec{E}_r^* \right) \right] \]

Poynting vector of incident wave.  

Poynting vector produced by the interference of incident and reflected waves.  

This becomes zero for lossless media.
Poynting Vectors in Medium 1

Poynting Vector of Incident Wave
\[ \vec{\phi}_i = \frac{1}{\omega \mu_i} \left[ \vec{E}_i \times (\vec{k}_i^* \times \vec{E}_i^*) \right] = \frac{1}{\omega \mu_i} \left[ \vec{k}_i^* \left( \vec{E}_i \cdot \vec{E}_i^* \right) - \vec{E}_i^* \left( \vec{k}_i^* \cdot \vec{E}_i \right) \right] \]

Poynting Vector of Reflected Wave
\[ \vec{\phi}_r = \frac{1}{\omega \mu_i} \left[ \vec{E}_r \times (\vec{k}_r^* \times \vec{E}_r^*) \right] = \frac{1}{\omega \mu_i} \left[ \vec{k}_r^* \left( \vec{E}_r \cdot \vec{E}_r^* \right) - \vec{E}_r^* \left( \vec{k}_r^* \cdot \vec{E}_r \right) \right] \]

Poynting Vector of Cross Terms
\[ \vec{\phi}_c = \frac{1}{\omega \mu_i} \left[ \vec{E}_i \times (\vec{k}_r^* \times \vec{E}_r^*) + \vec{E}_r \times (\vec{k}_i^* \times \vec{E}_i^*) \right] \]
\[ = \frac{1}{\omega \mu_i} \left[ \vec{k}_r^* \left( \vec{E}_i \cdot \vec{E}_r^* \right) - \vec{E}_r^* \left( \vec{k}_r^* \cdot \vec{E}_i \right) + \vec{k}_i^* \left( \vec{E}_r \cdot \vec{E}_i^* \right) - \vec{E}_i^* \left( \vec{k}_i^* \cdot \vec{E}_r \right) \right] \]

TM Analysis (1 of 6)

For a TM polarized wave in the x-z plane,
\[ \vec{E}_i = E_{0,i} \left( \cos \phi \hat{a}_x - \sin \phi \hat{a}_z \right) e^{-jkz} \]
\[ \vec{H}_i = \left( \tilde{E}_{0,i} / \eta_i \right) \hat{a}_x e^{-jkz} \]
\[ \vec{E}_r = E_{0,r} \left( \cos \phi \hat{a}_x + \sin \phi \hat{a}_z \right) e^{-jkz} \]
\[ \vec{H}_r = -\left( \tilde{E}_{0,r} / \eta_i \right) \hat{a}_z e^{-jkz} \]
\[ \vec{E}_t = E_{0,t} \left( \cos \phi \hat{a}_x - \sin \phi \hat{a}_z \right) e^{-jkz} \]
\[ \vec{H}_t = \left( \tilde{E}_{0,t} / \eta_i \right) \hat{a}_x e^{-jkz} \]
\[ \vec{k}_i = k_0 \hat{n}_i \left( \sin \phi \hat{a}_z + \cos \phi \hat{a}_x \right) \]
\[ \vec{k}_r = k_0 \hat{n}_r \left( \sin \phi \hat{a}_z - \cos \phi \hat{a}_x \right) \]
\[ \vec{k}_t = k_0 \hat{n}_t \left( \sin \phi \hat{a}_z + \cos \phi \hat{a}_x \right) \]
**TM Analysis (2 of 6)**

The $z$-component of the Poynting vectors at $z = 0$ are

$$\tilde{\phi}_{z,j} = \frac{|E_{0,j}|^2}{\eta_i^*} \cos \tilde{\theta}_i^*$$

$$\tilde{\phi}_{z,r} = -\frac{|E_{0,r}|^2}{\eta_i^*} \cos \tilde{\theta}_i^*$$

$$\tilde{\phi}_{z,c} = 2 \frac{\text{Im}[E_{0,j}^* E_{0,r}]}{\eta_i^*} \cos \tilde{\theta}_i^*$$

$$\tilde{\phi}_{z,t} = \frac{|E_{0,t}|^2}{\eta_i^*} \cos \tilde{\theta}_t^*$$

---

**TM Analysis (3 of 6)**

Power conservation is therefore

$$\tilde{\phi}_{z,1} = \tilde{\phi}_{z,2}$$

$$\tilde{\phi}_{z,j} + \tilde{\phi}_{z,r} + \tilde{\phi}_{z,c} = \tilde{\phi}_{z,t}$$

$$\tilde{\phi}_{z,j} + \tilde{\phi}_{z,r} + \tilde{\phi}_{z,c} = \tilde{\phi}_{z,t}$$

$$\frac{|E_{0,j}|^2}{\eta_i^*} \cos \tilde{\theta}_i^* - \frac{|E_{0,r}|^2}{\eta_i^*} \cos \tilde{\theta}_i^* + 2 \frac{\text{Im}[E_{0,j}^* E_{0,r}]}{\eta_i^*} \cos \tilde{\theta}_i^* = \frac{|E_{0,t}|^2}{\eta_i^*} \cos \tilde{\theta}_t^*$$

$$|\tilde{E}_{0,j}|^2 - |\tilde{E}_{0,r}|^2 + 2 \text{Im}[\tilde{E}_{0,j}^* \tilde{E}_{0,r}] = |\tilde{E}_{0,t}|^2 \eta_i^* \cos \tilde{\theta}_t^*$$
Recall that

\[ E_{0,r} = r_{TM} E_{0,i} \quad \text{and} \quad E_{0,i} = t_{TM} E_{0,i} \]

The power conservation equation becomes

\[ |E_{0,i}|^2 - |E_{0,r}|^2 + 2 \text{Im}[\bar{E}_{0,i} E_{0,r}] = |E_{0,i}|^2 \frac{\eta_i^* \cos \tilde{\theta}_i^*}{\eta_i^* \cos \tilde{\theta}_i^*} \]

\[ |E_{0,i}|^2 - |r_{TM} E_{0,i}|^2 + 2 \text{Im}[\bar{E}_{0,i} r_{TM} E_{0,i}] = |t_{TM} E_{0,i}|^2 \frac{\eta_i^* \cos \tilde{\theta}_i^*}{\eta_i^* \cos \tilde{\theta}_i^*} \]

\[ 1 - |r_{TM}|^2 + 2 \text{Im}[r_{TM}] = |t_{TM}|^2 \frac{\eta_i^* \cos \tilde{\theta}_i^*}{\eta_i^* \cos \tilde{\theta}_i^*} \]

\[ |r_{TM}|^2 - 2 \text{Im}[r_{TM}] + |t_{TM}|^2 \frac{\eta_i^* \cos \tilde{\theta}_i^*}{\eta_i^* \cos \tilde{\theta}_i^*} = 1 \]

Last, only the real part is retained to obtain reflectance \( R_{TM} \) and transmittance \( T_{TM} \).

\[ \text{Re} \left[ |r_{TM}|^2 - 2 \text{Im}[r_{TM}] + |t_{TM}|^2 \frac{\eta_i^* \cos \tilde{\theta}_i^*}{\eta_i^* \cos \tilde{\theta}_i^*} = 1 \right] \]

\[ \text{Re}[|r_{TM}|^2] - \text{Re}[2 \text{Im}[r_{TM}]] + \text{Re}[|t_{TM}|^2 \frac{\eta_i^* \cos \tilde{\theta}_i^*}{\eta_i^* \cos \tilde{\theta}_i^*}] = \text{Re}[1] \]

\[ |r_{TM}|^2 - 2 \text{Im}[r_{TM}] + |t_{TM}|^2 \text{Re}[\frac{\eta_i^* \cos \tilde{\theta}_i^*}{\eta_i^* \cos \tilde{\theta}_i^*}] = 1 \]
The power conservation equation is rearranged.

\[ |r_{TM}|^2 + |t_{TM}|^2 \text{Re} \left[ \frac{\eta_i \cos \Theta_i}{\eta_t \cos \Theta_t} \right] - 2 \text{Im}[r_{TM}] = 1 \]

Reflectance

\[ R_{TM} = |r_{TM}|^2 \]

Transmittance (sort of)

\[ T_{TM} = |t_{TM}|^2 \text{Re} \left[ \frac{\eta_i \cos \Theta_i}{\eta_t \cos \Theta_t} \right] \]

Cross Term

\[ \Delta_{TM} = 2 \text{Im}[r_{TM}] \]

* Clearly this is zero for lossless materials due to \( \text{Im}[\ ] \) operation.

Conservation Equation for Lossy Media

\[ R_{TM} + T_{TM} - \Delta_{TM} = 1 \]

Transmittance (actual)

\[ \tau_{TM} = T_{TM} - \Delta_{TM} \]

Summary for TM Polarization

Fresnel Equations

\[ \vec{E}_{TM} = \frac{\vec{E}_{TM}}{\vec{F}_{TM}} \]

Power Flow

Conservation:

\[ R_{TM} + T_{TM} - \Delta_{TM} = 1 \]

Reflectance (Lossy + Lossless):

\[ R_{TM} = |r_{TM}|^2 \]

Transmittance (Lossy):

\[ \tau_{TM} = T_{TM} - \Delta_{TM} \]

Transmittance (Lossless):

\[ T_{TM} = |t_{TM}|^2 \text{Re} \left[ \frac{\eta_i \cos \Theta_i}{\eta_t \cos \Theta_t} \right] \]

Cross Term (Lossy):

\[ \Delta_{TM} = 2 \text{Im}[r_{TM}] \]
Summary for TE Polarization

Fresnel Equations

\[ \tilde{r}_{TE} = \frac{E_{o,i}}{E_{o,j}} = \frac{\eta_i \cos \hat{\theta}_l - \eta_i \cos \hat{\theta}_r}{\eta_i \cos \hat{\theta}_l + \eta_i \cos \hat{\theta}_r} \]

\[ \tilde{r}_{TE} = \frac{\tilde{E}_{o,i}}{\tilde{E}_{o,j}} = \frac{2\tilde{\eta}_i \cos \hat{\theta}_l}{\eta_i \cos \hat{\theta}_l + \eta_i \cos \hat{\theta}_r} \]

Power Flow

Conservation:

\[ R_{TE} + T_{TE} - \Delta_{TE} = 1 \]

Reflectance (Lossy + Lossless):

\[ R_{TE} = |\tilde{r}_{TE}|^2 \]

Transmittance (Lossy):

\[ \tau_{TE} = T_{TE} - \Delta_{TE} \]

Transmittance (Lossless):

\[ T_{TE} = |\tilde{r}_{TE}|^2 \Re \left[ \frac{\eta_i \cos \hat{\theta}_l}{\eta_i \cos \hat{\theta}_r} \right] \]

Cross Term (Lossy):

\[ \Delta_{TM} = 2 \Im \left[ \tilde{r}_{TM} \right] \]

Relation Between the Parameters

For lossless materials, the transmittance reduces to

\[ T = |t|^2 \frac{\eta_i \cos \theta_l}{\eta_i \cos \theta_r} \]

It becomes possible to derive a relation between \( T_{TE} \) and \( T_{TM} \) for lossless materials.

\[ \frac{T_{TE}}{T_{TM}} = \frac{|t_{TM}|^2}{|t_{TM}|^2} = \frac{|t_{TM}|^2}{|t_{TM}|^2} = \frac{T_{TE}}{T_{TM}} = \frac{|t_{TM}|^2}{|t_{TM}|^2} \]
Total Reflectance and Transmittance

A wave incident onto a surface may have both TE and TM components. Power in the source wave is therefore

\[ P_{\text{inc}} = P_{\text{TE,inc}} + P_{\text{TM,inc}} \]

It follows that the total power reflected and transmitted is

\[ P_{\text{ref}} = R_{\text{TE}} P_{\text{TE,inc}} + R_{\text{TM}} P_{\text{TM,inc}} \quad P_{\text{tm}} = T_{\text{TE}} P_{\text{TE,inc}} + T_{\text{TM}} P_{\text{TM,inc}} \]

Overall reflectance \( R \) and transmittance \( T \) are derived by dividing these equations by the first equation.

\[
R = \frac{P_{\text{ref}}}{P_{\text{inc}}} = \frac{R_{\text{TE}} P_{\text{TE,inc}} + R_{\text{TM}} P_{\text{TM,inc}}}{P_{\text{TE,inc}} + P_{\text{TM,inc}}} \\
T = \frac{P_{\text{tm}}}{P_{\text{inc}}} = \frac{T_{\text{TE}} P_{\text{TE,inc}} + T_{\text{TM}} P_{\text{TM,inc}}}{P_{\text{TE,inc}} + P_{\text{TM,inc}}} 
\]

Example #5 – Fresnel Equations

In spherical coordinates, a wave is incident at \( \theta = 30^\circ \) and \( \phi = 120^\circ \) from air into water. Calculate the reflectance and transmittance for both TE and TM polarizations.

Solution

The reflectance for the TE polarization is

\[ R_{\text{TE}} = |r_{\text{TE}}|^2 = |-0.1749|^2 = 0.0306 = 3.1\% \]

The reflectance for the TM polarization is

\[ R_{\text{TM}} = |r_{\text{TM}}|^2 = |-0.1080|^2 = 0.0117 = 1.2\% \]
**Example #5 – Fresnel Equations**

In spherical coordinates, a wave is incident at $\theta = 30^\circ$ and $\phi = 120^\circ$ from air into water. Calculate the reflectance and transmittance for both TE and TM polarizations.

**Solution, cont’d**

The transmittance for the TE polarization is

$$T_{TE} = \left| r_{TE} \right|^2 \frac{\eta_1 \cos \theta}{\eta_2 \cos \theta_1} = \left| 0.825 \right|^2 \frac{376.73 \cos 22^\circ}{283.26 \cos 30^\circ} = 0.9694 = 96.9\%$$

The transmittance for the TM polarization is

$$T_{TM} = \left| r_{TM} \right|^2 \frac{\eta_1 \cos \theta}{\eta_2 \cos \theta_1} = \left| 0.8332 \right|^2 \frac{376.73 \cos 22^\circ}{283.26 \cos 30^\circ} = 0.9885 = 98.8\%$$

**Example #6 – Power Conservation**

In spherical coordinates, a wave is incident at $\theta = 30^\circ$ and $\phi = 120^\circ$ from air into water. Confirm power conservation for reflectance and transmittance of the TE and TM polarizations.

**Solution**

Power conservation for the TE polarization is

$$R_{TE} + T_{TE} = 3.1\% + 96.9\% = 100\%$$

Power conservation for the TM polarization is

$$R_{TM} + T_{TM} = 1.2\% + 98.8\% = 100\%$$
Example – Plot of Fresnel Equations

Plots of the Fresnel Equations

Low to High Index
\((n_1 = 1.0 \text{ and } n_2 = 1.5)\)

High to Low Index
\((n_1 = 1.5 \text{ and } n_2 = 1.0)\)