EE 4347
Applied Electromagnetics

Topic 3h

Scattering at an Interface:
Phase Matching & Special Angles

Lecture Outline

• Phase Matching at an Interface
• The Critical Angle
• Brewster’s Angle
The dispersion relation for isotropic materials is essentially just the Pythagorean theorem. It says a wave sees the same refractive index no matter what direction the wave is travelling.
Index Ellipsoid in Two Different Materials

Material 1 (Low $n$)
$$k_{x,1}^2 + k_{z,1}^2 = |\vec{k}_1|^2 = (k_\omega n_1)^2$$

Material 2 (High $n$)
$$k_{x,2}^2 + k_{z,2}^2 = |\vec{k}_2|^2 = (k_\omega n_2)^2$$

$n_1 < n_2$

Phase Matching When $n_1 < n_2$

Material 1
$$k_{x,1}^2 + k_{z,1}^2 = |\vec{k}_1|^2 = (k_\omega n_1)^2$$

Material 2
$$k_{x,2}^2 + k_{z,2}^2 = |\vec{k}_2|^2 = (k_\omega n_2)^2$$
**Summary of Phase Matching for** \( n_1 < n_2 \)

\[
\begin{align*}
 n_1 &< n_2 \\
 k_{x_1}^2 + k_{z_1}^2 &= |k_1|^2 = (k_0 n_1)^2 \\
 k_{x_2}^2 + k_{z_2}^2 &= |k_2|^2 = (k_0 n_2)^2
\end{align*}
\]

Properly phased matched at the interface.

**Phase Matching When** \( n_1 > n_2 \)

\[
\begin{align*}
 n_1 &> n_2 \\
 \theta_{inc} &< \theta_c \\
 k_{x_1}^2 + k_{z_1}^2 &= |k_1|^2 = (k_0 n_1)^2 \\
 \theta_{inc} &> \theta_c \\
 k_{x_2}^2 + k_{z_2}^2 &= |k_2|^2 = (k_0 n_2)^2
\end{align*}
\]
Summary of Phase Matching for $n_1 > n_2$

For $n_1 > n_2$,

For Material 1:

$$k_{x,1}^2 + k_{z,1}^2 = |k_1|^2 = \left(n_1 k_0\right)^2$$

For Material 2:

$$k_{x,2}^2 + k_{z,2}^2 = |k_2|^2 = \left(n_2 k_0\right)^2$$

1. $\theta_{inc} < \theta_c$ for $\theta_{inc} < \theta_c$
2. $\theta_{inc} < \theta_c$ for $\theta_{inc} > \theta_c$
3. $\theta_{inc} = \theta_c$ for $\theta_{inc} = \theta_c$
4. $\theta_{inc} > \theta_c$ for $\theta_{inc} > \theta_c$

Properly phased matched at the interface.

The Critical Angle
Phase Matching & Special Angles

Animation of Snell’s Law (1 of 2)

Beam propagates from low-index medium to a high-index medium.

\[ n_1 \sin \theta_i = n_2 \sin \theta_i \]

Animation of Snell’s Law (2 of 2)

Beam propagates from high-index medium to a low-index medium.

\[ n_1 \sin \theta_i = n_2 \sin \theta_i \]
The Critical Angle, $\theta_c$

The critical angle $\theta_c$ is the angle of incidence $\theta_i$ that produces an angle of transmission $\theta_t$ that is exactly 90°.

\[
n_i \sin \theta_i = n_2 \sin \theta_i
\]

\[
n_i \sin \theta_c = n_2 \sin (90°) \quad \Rightarrow \quad \theta_c = \sin^{-1}\left(\frac{n_2}{n_1} \sin (90°)\right)
\]

In order for there to be a critical angle $\theta_c$, the wave must be incident onto a low-index medium from a high-index medium.

\[
n_1 > n_2
\]

Field at an Interface Above and Below the Critical Angle (Ignoring Reflections)

1. The field always penetrates into material 2, but it may not propagate.
2. Above the critical angle, penetration is greatest near the critical angle.
3. Very high spatial frequencies are supported in material 2 despite the dispersion relation.
4. In material 2, power always flows in the transverse direction, but not necessarily in the longitudinal direction.

This called an “evanescent field”
Simulation of Reflection and Transmission at a Single Interface ($n_1 > n_2$)

$n_1 = 1.41$, $n_2 = 1.0 \rightarrow \theta_c = 45^\circ$

Field Visualization for $\theta_c = 45^\circ$

$\theta_{inc} = 44^\circ$  $\theta_{inc} = 46^\circ$

$\theta_{inc} = 67^\circ$  $\theta_{inc} = 89^\circ$
Electromagnetic Tunneling

If an evanescent field touches a medium with higher refractive index, the field may no longer be cutoff and become a propagating wave.

This is a very unusual phenomenon because the evanescent field is contributing to power flow.

This is called electromagnetic tunneling and is analogous to electron tunneling through thin insulators.

Brewster’s Angle
Can Reflection Ever Be Zero?

If we inspect the Fresnel equations long enough, we see that the reflection coefficients have a difference in their numerator. This means there must exist special conditions where reflection can be zero. These are the Brewster’s angles.

**TE, s, ⊥ Polarization**

\[
\begin{align*}
    r_{\text{TE}} &= \frac{\eta_2 \cos \theta - \eta_1 \cos \theta_1}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_1} \\
    t_{\text{TE}} &= \frac{2\eta_2 \cos \theta_1}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_1} \\
    1 + r_{\text{TE}} &= t_{\text{TE}}
\end{align*}
\]

**TM, p, ∥ Polarization**

\[
\begin{align*}
    r_{\text{TM}} &= \frac{\eta_2 \cos \theta - \eta_1 \cos \theta_1}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_1} \\
    t_{\text{TM}} &= \frac{2\eta_2 \cos \theta_1}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_1} \\
    1 + r_{\text{TM}} &= \frac{\cos \theta_1}{\cos \theta_1} t_{\text{TM}}
\end{align*}
\]

We do not observe a similar condition for transmission.

Brewster’s Angle for TE Polarization (1 of 3)

We start with the Fresnel equation for reflection of the TE polarization.

\[
r_{\text{TE}} = \frac{\eta_2 \cos \theta_1 - \eta_1 \cos \theta}{\eta_2 \cos \theta_1 + \eta_1 \cos \theta_1}
\]

We set the numerator equation to zero.

\[
\eta_2 \cos \theta_1 - \eta_1 \cos \theta_1 = 0
\]

The Brewster’s angle \( \theta_B \) is the angle of incidence \( \theta_1 \) that satisfies this expression and makes reflection go to zero.

\[
\begin{align*}
    \eta_2 \cos \theta_1 - \eta_1 \cos \theta_1 &= 0 \\
    \eta_2 \cos \theta_1 &= \eta_1 \cos \theta_1 \\
    \cos \theta_1 &= \frac{\eta_1}{\eta_2} \cos \theta_1 \\
    \cos^2 \theta_1 &= \left( \frac{\eta_1}{\eta_2} \right)^2 \cos^2 \theta_1 \\
    1 - \sin^2 \theta_1 &= \left( \frac{\eta_1}{\eta_2} \right)^2 - \left( \frac{\eta_1}{\eta_2} \right)^2 \sin^2 \theta_1
\end{align*}
\]

We would like to eliminate this term.
Brewster’s Angle for TE Polarization (2 of 3)

Solve Snell’s law for $\sin^2 \theta_t$.

$$n_t \sin \theta_B = n_2 \sin \theta_t \quad \rightarrow \quad \sin^2 \theta_t = \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_B$$

Substitute this result into our previous expression for the Brewster’s angle.

$$1 - \sin^2 \theta_B = \left( \frac{n_1}{n_2} \right)^2 - \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_t$$

$$\downarrow$$

$$1 - \sin^2 \theta_B = \left( \frac{n_1}{n_2} \right)^2 \left( \frac{n_1}{n_2} \right)^2 \sin^2 \theta_B \quad \rightarrow \quad \sin^2 \theta_{B,TE} = \frac{1 - (n_1/n_2)^2}{1 - (n_1n_2/n_2n_2)^2}$$

Brewster’s Angle for TE Polarization (3 of 3)

Now write $\eta$ and $n$ and in terms of $\mu$ and $\varepsilon$.

$$\eta_1 = \eta_0 \sqrt{\frac{\mu_1}{\varepsilon_1}} \quad \eta_2 = \eta_0 \sqrt{\frac{\mu_2}{\varepsilon_2}} \quad n_1 = \sqrt{\mu_1 \varepsilon_1} \quad n_2 = \sqrt{\mu_2 \varepsilon_2}$$

The expression for the Brewster’s angle becomes

$$\sin^2 \theta_{B,TE} = \frac{1 - \frac{\mu_1}{\mu_2} \frac{\varepsilon_{2,1}}{\varepsilon_{1,2}}}{1 - \left( \frac{\mu_1}{\mu_2} \right)^2}$$

Inspecting this equation, we see that there is no Brewster’s angle for the TE polarization unless the permeability is different in each medium.
Brewster’s Angle for TM Polarization (1 of 3)

We start with the Fresnel equation for reflection of the TM polarization.

\[
r_{\text{TM}} = \frac{n_2 \cos \theta_i - n_1 \cos \theta_i}{n_2 \cos \theta_i + n_1 \cos \theta_i}
\]

We set the numerator equation to zero.

\[
n_2 \cos \theta_i - n_1 \cos \theta_i = 0
\]

The Brewster’s angle \( \theta_B \) is the angle of incidence \( \theta_i \) that satisfies this expression and makes reflection go to zero.

\[
n_2 \cos \theta_i - n_1 \cos \theta_B = 0
\]

\[
1 - \sin^2 \theta_B = \left( \frac{n_2}{n_1} \right)^2 - \left( \frac{n_2}{n_1} \right)^2 \sin^2 \theta_i
\]

Brewster’s Angle for TM Polarization (2 of 3)

Solve Snell’s law for \( \sin^2 \theta_i \).

\[
n_i \sin \theta_i = n_2 \sin \theta \\ \rightarrow \sin^2 \theta_i = \left( \frac{n_i}{n_2} \right)^2 \sin^2 \theta_B
\]

Substitute this result into our previous expression for the Brewster’s angle.

\[
1 - \sin^2 \theta_B = \left( \frac{n_2}{n_1} \right)^2 - \left( \frac{n_2}{n_1} \right)^2 \sin^2 \theta_i
\]

\[
1 - \sin^2 \theta_B = \left( \frac{n_2}{n_1} \right)^2 - \left( \frac{n_2}{n_1} \right)^2 \left( \frac{n_i}{n_2} \right)^2 \sin^2 \theta_B \rightarrow \sin^2 \theta_{B,\text{TM}} = \frac{1 - \left( \frac{n_2}{n_1} \right)^2}{1 - \left( n_1 n_2 / n_2 n_1 \right)^2}
\]
We now write the impedances and refractive indices in terms of

\[ \eta_1 = \eta_0 \sqrt{\frac{\mu_1}{\varepsilon_{r1}}} \quad \eta_2 = \eta_0 \sqrt{\frac{\mu_2}{\varepsilon_{r2}}} \quad n_1 = \sqrt{\mu_1\varepsilon_{r1}} \quad n_2 = \sqrt{\mu_2\varepsilon_{r2}} \]

Our expression for the Brewster’s angle becomes

\[ \sin^2 \theta_{B,TM} = \frac{1 - \frac{\mu_2}{\varepsilon_{r1}}}{1 - \left( \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \right)^2} \]

Inspecting this equation, we see that we still have a Brewster’s angle even when the materials do not have a magnetic response.

\[ \tan \theta_{B,TM} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} = \frac{n_2}{n_1} \]

\[ \mu_1 = \mu_2 = 1 \]

Simulation of Reflection and Transmission at a Single Interface \((n_1<n_2)\)

\(n_1=1.0, n_2=1.73 \rightarrow \theta_B=60^\circ\)
Example – Plot of Fresnel Equations

Plots of the Fresnel Equations

Low to High Index
\( (n_1 = 1.0 \text{ and } n_2 = 1.5) \)

No critical angle
\[ \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}(1.5) \]

Brewster’s angle
\[ \theta_{B,\text{TM}} = \tan^{-1}\left(\frac{n_1}{n_2}\right) = \tan^{-1}(1.5) = 56.31^\circ \]

High to Low Index
\( (n_1 = 1.5 \text{ and } n_2 = 1.0) \)

Critical angle
\[ \theta_c = \sin^{-1}\left(\frac{n_1}{n_2}\right) = \sin^{-1}(0.667) = 41.81^\circ \]

Brewster’s angle
\[ \theta_{B,\text{TM}} = \tan^{-1}\left(\frac{n_1}{n_2}\right) = \tan^{-1}(0.667) = 33.69^\circ \]