EE 4347
Applied Electromagnetics

Topic 4a

Transmission Line Equations

Lecture Outline

• Introduction
• Transmission Line Equations
• Transmission Line Wave Equations
Introduction
Transmission Line Equations

Transmission Line Parameters RLGC

We can think of transmission lines as being composed of millions of tiny little circuit elements that are distributed along the length of the line.

In fact, these circuit elements are not discrete, but continuous along the length of the transmission line.
It is not technically correct to represent a transmission line with discrete circuit elements like this. However, if the size of the circuit $\Delta z$ is very small compared to the wavelength of the signal on the transmission line, it becomes an accurate and effective way to model the transmission line.

There are many possible circuit models for transmission lines, but most produce the same equations after analysis.
L-Type Equivalent Circuit Model

**Distributed Circuit Parameters**

- **$R$ ($\Omega/m$)**
  - Resistance per unit length.
  - Arises due to resistivity in the conductors.

- **$L$ (H/m)**
  - Inductance per unit length.
  - Arises due to stored magnetic energy around the line.

- **$G$ (1/Ω·m)**
  - Conductance per unit length.
  - Arises due to conductivity in the dielectric separating the conductors.
  
  \[ G \neq \frac{1}{R} \]

- **$C$ (F/m)**
  - Capacitance per unit length.
  - Arises due to stored electric energy between the conductors.

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Relation to Electromagnetic Parameters

Every transmission line with a homogeneous fill has:

\[ LC = \mu \varepsilon \]

\[ G = \frac{\sigma}{C} = \frac{\varepsilon}{\varepsilon} \]
Example RLGc Parameters

<table>
<thead>
<tr>
<th>RG-59 Coax</th>
<th>CAT5 Twisted Pair</th>
<th>Microstrip</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = 36 \ \text{mΩ/m}$</td>
<td>$R = 176 \ \text{mΩ/m}$</td>
<td>$R = 150 \ \text{mΩ/m}$</td>
</tr>
<tr>
<td>$L = 430 \ \text{nH/m}$</td>
<td>$L = 490 \ \text{nH/m}$</td>
<td>$L = 364 \ \text{nH/m}$</td>
</tr>
<tr>
<td>$G = 10 \ \mu\Omega/m$</td>
<td>$G = 2 \ \mu\Omega/m$</td>
<td>$G = 3 \ \mu\Omega/m$</td>
</tr>
<tr>
<td>$C = 69 \ \text{pF/m}$</td>
<td>$C = 49 \ \text{pF/m}$</td>
<td>$C = 107 \ \text{pF/m}$</td>
</tr>
<tr>
<td>$Z_0 = 75 \ \Omega$</td>
<td>$Z_0 = 100 \ \Omega$</td>
<td>$Z_0 = 50 \ \Omega$</td>
</tr>
</tbody>
</table>

Surprisingly, almost all transmission lines have parameters very close to these same values.
**E & H → V and I**

Fundamentally, all circuit problems are electromagnetic problems and can be solved as such.

All two-conductor transmission lines either support a TEM wave or a wave very closely approximated as TEM.

An important property of TEM waves is that $E$ is uniquely related to $V$ and $H$ and uniquely related to $E$.

$$V = -\int_L \vec{E} \cdot d\ell$$

$$I = \oint_L \vec{H} \cdot d\ell$$

This lets us analyze transmission lines in terms of just $V$ and $I$. This makes analysis much simpler because these are scalar quantities!

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**Transmission Line Equations**

The transmission line equations do for transmission lines the same thing as Maxwell’s curl equations do for waves.

Maxwell’s Equations

\[
\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \\
\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}
\]

Transmission Line Equations

\[
-\frac{\partial V}{\partial z} = RI + L \frac{\partial I}{\partial t} \\
-\frac{\partial I}{\partial z} = GV + C \frac{\partial V}{\partial t}
\]

Like Maxwell’s equations, the transmission line equations are rarely directly useful. Instead, we will derive all of the useful equations from them.
Derivation of First TL Equation (1 of 2)

Apply Kirchoff’s voltage law (KVL) to the outer loop of the equivalent circuit:

\[-V(z,t) + I(z,t)R\Delta z + L\Delta z \frac{\partial I(z,t)}{\partial t} + V(z + \Delta z,t) = 0\]

Derivation of First TL Equation (2 of 2)

We rearrange the equation by bringing all of the voltage terms to the left-hand side of the equation, bringing all of the current terms to the right-hand side of the equation, and then dividing both sides by \(\Delta z\).

\[-V(z,t) + I(z,t) R\Delta z + L\Delta z \frac{\partial I(z,t)}{\partial t} + V(z + \Delta z,t) = 0\]

\[-\frac{V(z + \Delta z,t) - V(z,t)}{\Delta z} = RI(z,t) + L \frac{\partial I(z,t)}{\partial t}\]

In the limit as \(\Delta z \rightarrow 0\), the expression on the left-hand side becomes a derivative with respect to \(z\).

\[-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L \frac{\partial I(z,t)}{\partial t}\]
Derivation of Second TL Equation (1 of 2)

Apply Kirchhoff's current law (KCL) to the main node the equivalent circuit:

\[
I(z,t) - I(z + \Delta z,t) - G\Delta z V(z + \Delta z, t) - C\Delta z \frac{\partial V(z + \Delta z, t)}{\partial t} = 0
\]

In the limit as \(\Delta z \to 0\), the expression on the left-hand side becomes a derivative with respect to \(z\).

\[
\frac{\partial I(z,t)}{\partial z} = GV(z,t) + C \frac{\partial V(z,t)}{\partial t}
\]
Transmission Line
Wave Equations

Starting Point – Telegrapher Equations

Start with the transmission line equations derived in the previous section.

\[
\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L\frac{\partial I(z,t)}{\partial t} \quad \frac{\partial I(z,t)}{\partial z} = GV(z,t) + C\frac{\partial V(z,t)}{\partial t}
\]

time-domain

For time-harmonic (i.e. frequency-domain) analysis, Fourier transform the equations above.

\[
\frac{dV(z)}{dz} = (R + j\omega L)I(z) \quad \frac{dI(z)}{dz} = (G + j\omega C)V(z)
\]

frequency-domain

Note: The derivative \(d/dz\) became an ordinary derivative because \(z\) is the only independent variable left.

These last equations are commonly referred to as the *telegrapher equations*.
Wave Equation in Terms of $V(z)$

\[-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \quad \text{Eq. (1)} \quad -\frac{dI(z)}{dz} = (G + j\omega C)V(z) \quad \text{Eq. (2)}\]

To derive a wave equation in terms of $V(z)$, first differentiate Eq. (1) with respect to $z$.

\[-\frac{d^2V(z)}{dz^2} = (R + j\omega L)\frac{dI(z)}{dz} \quad \text{Eq. (3)}\]

Second, substitute Eq. (2) into the right-hand side of Eq. (3) to eliminate $I(z)$ from the equation.

\[-\frac{d^2V(z)}{dz^2} = -(R + j\omega L)(G + j\omega C)V(z)\]

Last, rearrange the terms to arrive at the final form of the wave equation.

\[-\frac{d^2V(z)}{dz^2} - (R + j\omega L)(G + j\omega C)V(z) = 0\]

Wave Equation in Terms of $I(z)$

\[-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \quad \text{Eq. (1)} \quad -\frac{dI(z)}{dz} = (G + j\omega C)V(z) \quad \text{Eq. (2)}\]

To derive a wave equation in terms of just $I(z)$, first differentiate Eq. (2) with respect to $z$.

\[-\frac{d^2I(z)}{dz^2} = (G + j\omega C)\frac{dV(z)}{dz} \quad \text{Eq. (3)}\]

Second, substitute Eq. (1) into the right-hand side of Eq. (3) to eliminate $V(z)$ from the equation.

\[-\frac{d^2I(z)}{dz^2} = -(G + j\omega C)(R + j\omega L)I(z)\]

Last, rearrange the terms to arrive at the final form of the wave equation.

\[-\frac{d^2I(z)}{dz^2} - (G + j\omega C)(R + j\omega L)I(z) = 0\]
Propagation Constant, $\gamma$

In our wave equations, there is the common term $(G + j\omega C)(R + j\omega L)$.

Define the propagation constant $\gamma$ to be

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Given this definition, the transmission line equations are written as

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

Solution to the Wave Equations

If the wave equations are handed off to a mathematician, they will return with the following solutions.

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \quad \rightarrow \quad V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \quad \rightarrow \quad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

Both $V(z)$ and $I(z)$ have the same differential equation so it makes sense they have the same solution.