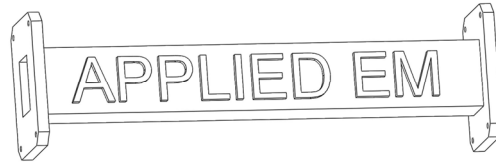




**Course Instructor**  
 Dr. Raymond C. Rumpf  
 Office: A-337  
 Phone: (915) 747-6958  
 E-Mail: rcrumpf@utep.edu



EE 4347

## Applied Electromagnetics

Topic 4a

# Transmission Lines

◆◆ These notes may contain copyrighted material obtained under fair use rules. Distribution of these materials is strictly prohibited ◆◆

## Lecture Outline



- Introduction
- Transmission Line Equations
- Transmission Line Wave Equations
- Transmission Line Parameters
  - $\alpha$  and  $\beta$
  - Characteristic Impedance,  $Z_0$
- Special Cases of Transmission Lines
  - General transmission lines
  - Lossless lines
  - Weakly absorbing lines
  - Distortionless lines
- Examples
  - RG-59 coaxial cable
  - Microstrip design

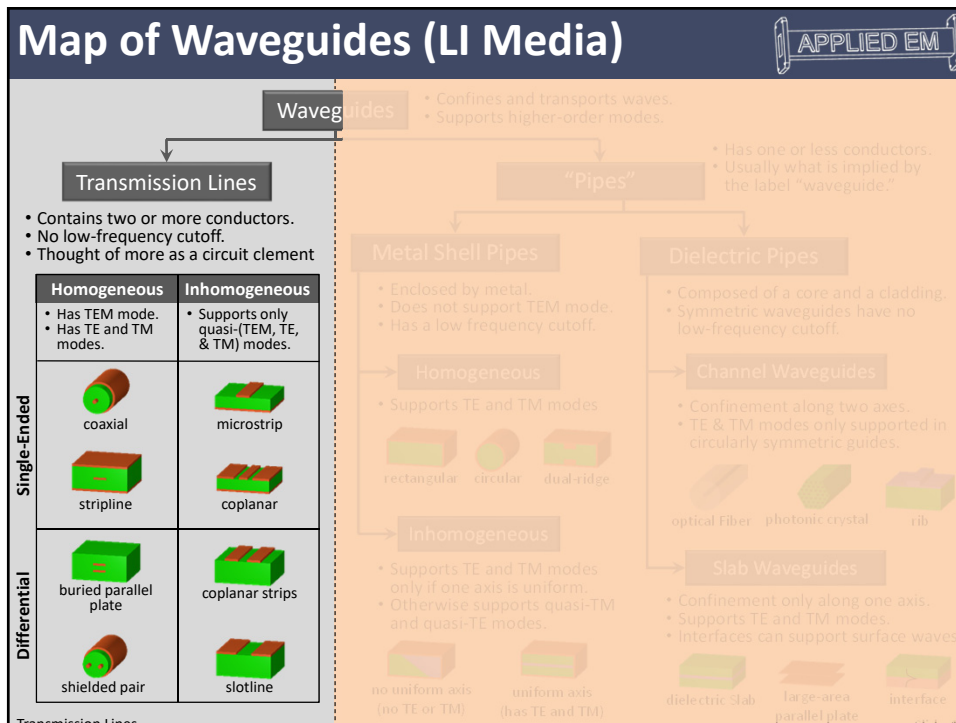
Transmission Lines

Slide 2

# Introduction

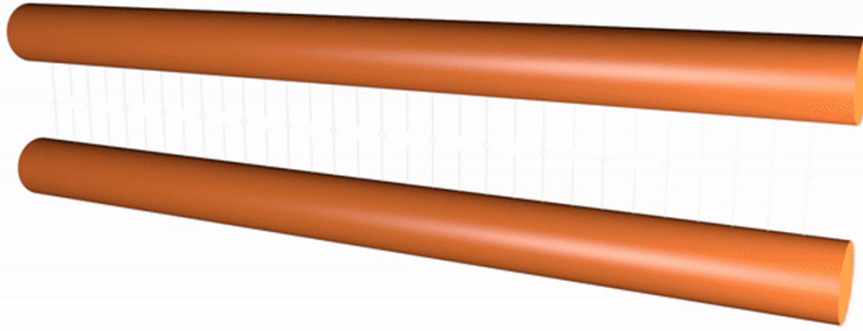
Transmission Lines

Slide 3



## Transmission Line Parameters RLGC APPLIED EM

We can think transmission lines as being composed of millions of tiny little circuit elements that are distributed along the length of the line.



In fact, these circuit element are not discrete, but continuous along the length of the transmission line.

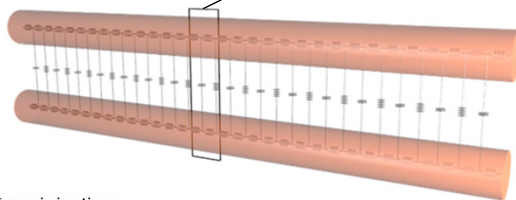
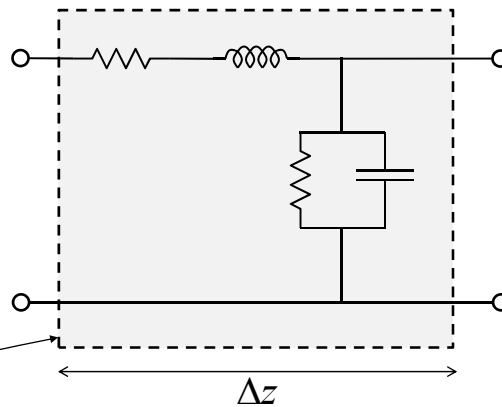
Transmission Lines

Slide 5

## RLGC Circuit Model APPLIED EM

It is not technically correct to represent a transmission line with discrete circuit elements like this.

However, if the size of the circuit  $\Delta z$  is very small compared to the wavelength of the signal on the transmission line, it becomes an accurate and effective way to model the transmission line.



Transmission Lines

Slide 6

## L-Type Equivalent Circuit Model APPLIED EM

**Distributed Circuit Parameters**

$R$  ( $\Omega/m$ )  
Resistance per unit length.  
Arises due to resistivity in the conductors.

$L$  (H/m)  
Inductance per unit length.  
Arises due to stored magnetic energy around the line.

$G$  ( $1/\Omega \cdot m$ )  
Conductance per unit length.  
Arises due to conductivity in the dielectric separating the conductors.

$$G \neq \frac{1}{R}$$

$C$  (F/m)  
Capacitance per unit length.  
Arises due to stored electric energy between the conductors.

There are many possible circuit models for transmission lines, but most produce the same equations after analysis.

$z$   $z + \Delta z$

Transmission LinesSlide 7

## Relation to Electromagnetic Parameters APPLIED EM

Every transmission line with a homogeneous fill has:

$$LC = \mu\epsilon$$

$$\frac{G}{C} = \frac{\sigma}{\epsilon}$$

Transmission LinesSlide 8

## Fundamental Vs. Intuitive Parameters APPLIED EM

Fundamental Parameters	Intuitive Parameters
Electromagnetics $\mu, \epsilon, \sigma$	Electromagnetics $n, \eta, \alpha, \beta, \tan \delta$
Transmission Lines $R, L, G, C$	Transmission Lines $Z_0, \alpha, \beta, \text{VSWR}$

The fundamental parameters are the most basic parameters needed to solve a transmission line problem.

However, it is difficult to be intuitive about how they affect signals on the line.

An electromagnetic analysis is needed to determine  $R, L, G,$  and  $C$  from the geometry of the transmission line.

The intuitive parameters provide intuitive insight about how signals behave on a transmission line.

They isolate specific information to a single parameter.

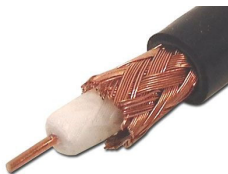
The intuitive parameters are calculated from  $R, L, G,$  and  $C$ .

Transmission Lines

Slide 9

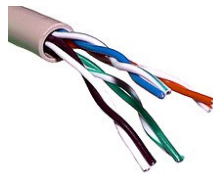
## Example RLGC Parameters APPLIED EM

**RG-59 Coax**



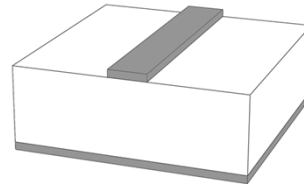
$R = 36 \text{ m}\Omega/\text{m}$   
 $L = 430 \text{ nH}/\text{m}$   
 $G = 10 \text{ }\mu\text{S}/\text{m}$   
 $C = 69 \text{ pF}/\text{m}$   
 $Z_0 = 75 \text{ }\Omega$

**CAT5 Twisted Pair**



$R = 176 \text{ m}\Omega/\text{m}$   
 $L = 490 \text{ nH}/\text{m}$   
 $G = 2 \text{ }\mu\text{S}/\text{m}$   
 $C = 49 \text{ pF}/\text{m}$   
 $Z_0 = 100 \text{ }\Omega$

**Microstrip**



$R = 150 \text{ m}\Omega/\text{m}$   
 $L = 364 \text{ nH}/\text{m}$   
 $G = 3 \text{ }\mu\text{S}/\text{m}$   
 $C = 107 \text{ pF}/\text{m}$   
 $Z_0 = 50 \text{ }\Omega$

Surprisingly, almost all transmission lines have parameters very close to these same values.

Transmission Lines

Slide 10

# Transmission Line Equations

Transmission Lines

Slide 11

## $E$ & $H \rightarrow V$ and $I$



Fundamentally, all circuit problems are electromagnetic problems and can be solved as such.

All two-conductor transmission lines either support a TEM wave or a wave very closely approximated as TEM.

An important property of TEM waves is that  $E$  is uniquely related to  $V$  and  $H$  and uniquely related to  $I$ .

$$V = -\int_L \vec{E} \cdot d\vec{\ell} \qquad I = \oint_L \vec{H} \cdot d\vec{\ell}$$

This let's us analyze transmission lines in terms of just  $V$  and  $I$ . This makes analysis much simpler because these are scalar quantities!

Transmission Lines

Slide 12

## Transmission Line Equations



The transmission line equations do for transmission lines the same thing as Maxwell's curl equations do for unguided waves.

Maxwell's Equations

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Transmission Line Equations

$$-\frac{\partial V}{\partial z} = RI + L \frac{\partial I}{\partial t}$$

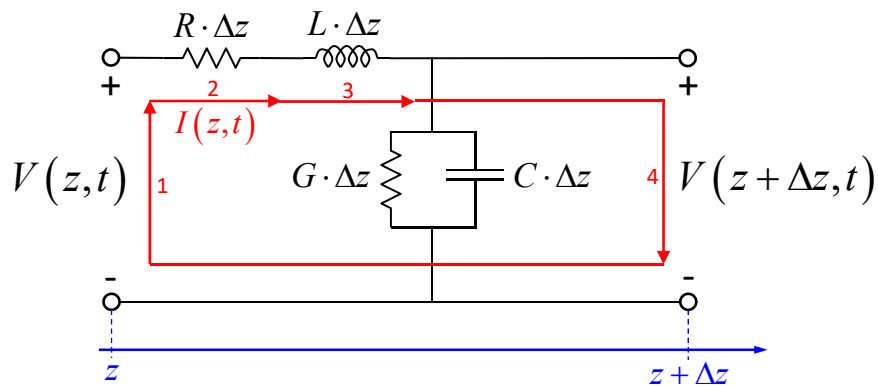
$$-\frac{\partial I}{\partial z} = GV + C \frac{\partial V}{\partial t}$$

Like Maxwell's equations, the transmission line equations are rarely directly useful. Instead, we will derive all of the useful equations from them.

Transmission Lines

Slide 13

## Derivation of First TL Equation (1 of 2)



Apply Kirchoff's voltage law (KVL) to the outer loop of the equivalent circuit:

$$-\underbrace{V(z, t)}_1 + \underbrace{I(z, t)R\Delta z}_2 + \underbrace{L\Delta z \frac{\partial I(z, t)}{\partial t}}_3 + \underbrace{V(z + \Delta z, t)}_4 = 0$$

Transmission Lines

Slide 14

## Derivation of First TL Equation (2 of 2) APPLIED EM

We rearrange the equation by bringing all of the voltage terms to the left-hand side of the equation, bringing all of the current terms to the right-hand side of the equation, and then dividing both sides by  $\Delta z$ .

$$-V(z,t) + I(z,t)R\Delta z + L\Delta z \frac{\partial I(z,t)}{\partial t} + V(z+\Delta z,t) = 0$$

↓

$$-\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = RI(z,t) + L \frac{\partial I(z,t)}{\partial t}$$

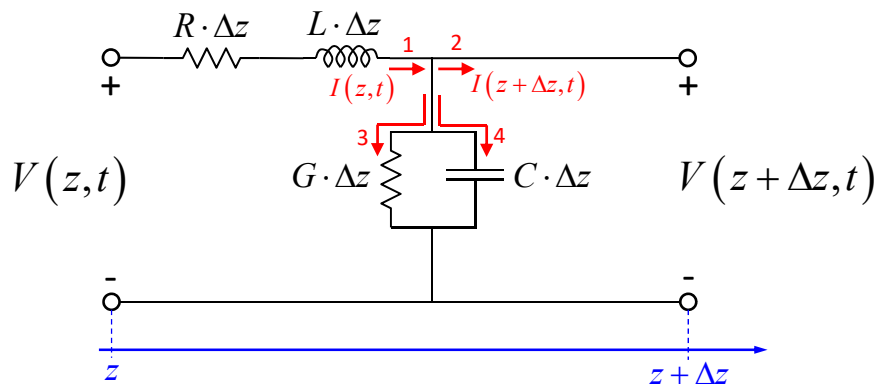
In the limit as  $\Delta z \rightarrow 0$ , the expression on the left-hand side becomes a derivative with respect to  $z$ .

$$\boxed{-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L \frac{\partial I(z,t)}{\partial t}}$$

Transmission Lines

Slide 15

## Derivation of Second TL Equation (1 of 2) APPLIED EM



Apply Kirchoff's current law (KCL) to the main node the equivalent circuit:

$$\underbrace{I(z,t)}_1 - \underbrace{I(z+\Delta z,t)}_2 - \underbrace{G\Delta z V(z+\Delta z,t)}_3 - \underbrace{C\Delta z \frac{\partial V(z+\Delta z,t)}{\partial t}}_4 = 0$$

Transmission Lines

Slide 16



## Derivation of Second TL Equation (2 of 2)



We rearrange the equation by bringing all of the current terms to the left-hand side of the equation, bringing all of the voltage terms to the right-hand side of the equation, and then dividing both sides by  $\Delta z$ .

$$I(z, t) - I(z + \Delta z, t) - G\Delta z V(z + \Delta z, t) - C\Delta z \frac{\partial V(z + \Delta z, t)}{\partial t} = 0$$

↓

$$-\frac{I(z + \Delta z, t) - I(z, t)}{\Delta z} = GV(z + \Delta z, t) + C \frac{\partial V(z + \Delta z, t)}{\partial t}$$

In the limit as  $\Delta z \rightarrow 0$ , the expression on the left-hand side becomes a derivative with respect to  $z$ .

$$\boxed{-\frac{\partial I(z, t)}{\partial z} = GV(z, t) + C \frac{\partial V(z, t)}{\partial t}}$$

Transmission Lines

Slide 17

# Transmission Line Wave Equations

Transmission Lines

Slide 18

## Starting Point – Telegrapher Equations



We start with the transmission line equations derived in the previous section.

$$-\frac{\partial V(z,t)}{\partial z} = RI(z,t) + L \frac{\partial I(z,t)}{\partial t} \quad -\frac{\partial I(z,t)}{\partial z} = GV(z,t) + C \frac{\partial V(z,t)}{\partial t} \quad \text{time-domain}$$

For time-harmonic (i.e. frequency-domain) analysis, we Fourier transform the equations above.

$$\boxed{-\frac{dV(z)}{dz} = (R + j\omega L)I(z)} \quad \boxed{-\frac{dI(z)}{dz} = (G + j\omega C)V(z)} \quad \text{frequency-domain}$$

Note: Our derivative  $d/dz$  became an ordinary derivative because  $z$  is the only independent variable left.

These last equations are commonly referred to as the *telegrapher equations*.

Transmission Lines

Slide 19

## Wave Equation in Terms of $V(z)$



$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \quad \text{Eq. (1)} \quad -\frac{dI(z)}{dz} = (G + j\omega C)V(z) \quad \text{Eq. (2)}$$

To derive a wave equation in terms of  $V(z)$ , we first differentiate Eq. (1) with respect to  $z$ .

$$-\frac{d^2V(z)}{dz^2} = (R + j\omega L) \frac{dI(z)}{dz} \quad \text{Eq. (3)}$$

Second, we substitute Eq. (2) into the right-hand side of Eq. (3) to eliminate  $I(z)$  from the equation.

$$-\frac{d^2V(z)}{dz^2} = -(R + j\omega L)(G + j\omega C)V(z)$$

Last, we rearrange the terms to arrive at the final form of the wave equation.

$$\boxed{\frac{d^2V(z)}{dz^2} - (R + j\omega L)(G + j\omega C)V(z) = 0}$$

Transmission Lines

Slide 20

## Wave Equation in Terms of $I(z)$



$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \quad \text{Eq. (1)}$$

$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z) \quad \text{Eq. (2)}$$

To derive a wave equation in terms of just  $I(z)$ , we first differentiate Eq. (2) with respect to  $z$ .

$$-\frac{d^2I(z)}{dz^2} = (G + j\omega C)\frac{dV(z)}{dz} \quad \text{Eq. (3)}$$

Second, we substitute Eq. (1) into the right-hand side of Eq. (3) to eliminate  $V(z)$  from the equation.

$$-\frac{d^2I(z)}{dz^2} = -(G + j\omega C)(R + j\omega L)I(z)$$

Last, we rearrange the terms to arrive at the final form of the wave equation.

$$\frac{d^2I(z)}{dz^2} - (G + j\omega C)(R + j\omega L)I(z) = 0$$

Transmission Lines

Slide 21

## Propagation Constant, $\gamma$



In our wave equations, we have a common term  $(G + j\omega C)(R + j\omega L)$ .

Define the propagation constant  $\gamma$  to be

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Given this definition, the transmission line equations are written as

$$\frac{d^2V(z)}{dz^2} - \gamma^2V(z) = 0$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2I(z) = 0$$

Transmission Lines

Slide 22

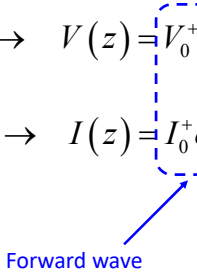
## Solution to the Wave Equations

APPLIED EM

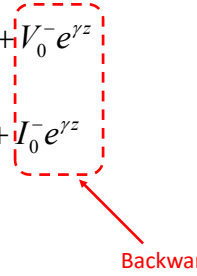
If we hand the wave equations off to a mathematician, they will return with the following solutions.

$$\frac{d^2V(z)}{dz^2} - \gamma^2V(z) = 0 \quad \rightarrow \quad V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2I(z) = 0 \quad \rightarrow \quad I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$



Forward wave



Backward wave

Both  $V(z)$  and  $I(z)$  have the same differential equation so it makes sense they have the same solution.

Transmission Lines
Slide 23

# Transmission Line Parameters:

Attenuation Coefficient,  $\alpha$   
Phase Constant,  $\beta$

## Derivation $\alpha$ and $\beta$ (1 of 7)



Step 1 – Start with our expression for  $\gamma$ .

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Square this expression to get rid of square-root on right-hand side.

$$(\alpha + j\beta)^2 = (G + j\omega C)(R + j\omega L)$$

Expand this expression.

$$\alpha^2 + j2\alpha\beta - \beta^2 = RG + j\omega RC + j\omega LG - \omega^2 LC$$

Collect real and imaginary parts on the left-hand and right-hand sides.

$$(\alpha^2 - \beta^2) + j2\alpha\beta = (RG - \omega^2 LC) + j\omega(RC + LG)$$

Transmission Lines

Slide 25

## Derivation $\alpha$ and $\beta$ (2 of 7)



Step 2 – Generate two equations by equating real and imaginary parts.

$$\begin{array}{c}
 2\alpha\beta = \omega(RC + LG) \\
 \swarrow \quad \searrow \\
 (\alpha^2 - \beta^2) + j2\alpha\beta = (RG - \omega^2 LC) + j\omega(RC + LG) \\
 \swarrow \quad \searrow \\
 \alpha^2 - \beta^2 = RG - \omega^2 LC
 \end{array}$$

We now have two equations and two unknowns.

Transmission Lines

Slide 26

## Derivation $\alpha$ and $\beta$ (3 of 7)



Step 3 – Derive a quadratic equation for  $\alpha^2$ .

$$2\alpha\beta = \omega(RC + LG) \quad \text{Eq. (1a)}$$

$$\alpha^2 - \beta^2 = RG - \omega^2 LC \quad \text{Eq. (1b)}$$

Solve Eq. (1a) for  $\beta$ .

$$\beta = \frac{\omega}{2\alpha}(RC + LG) \quad \text{Eq. (2)}$$

Substitute Eq. (2) into Eq. (1b) and simplify.

$$\alpha^2 - \left[ \frac{\omega}{2\alpha}(RC + LG) \right]^2 = RG - \omega^2 LC$$

$$\alpha^2 - \frac{\omega^2(RC + LG)^2}{4\alpha^2} = RG - \omega^2 LC$$

$$4\alpha^4 - \omega^2(RC + LG)^2 = 4\alpha^2 RG - 4\alpha^2 \omega^2 LC$$

$$\alpha^4 + \alpha^2(\omega^2 LC - RG) - \left[ \frac{\omega}{2}(RC + LG) \right]^2 = 0$$

Transmission Lines

Slide 27

## Derivation $\alpha$ and $\beta$ (4 of 7)



Step 4 – Solve for  $\alpha^2$  using the quadratic formula.

$$\text{Recall the quadratic formula: } ax^2 + bx + c = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Our equation for  $\alpha$  is in the form of a quadratic equation where

$$a = 1$$

$$b = \omega^2 LC - RG$$

$$\alpha^4 + \alpha^2(\omega^2 LC - RG) - \left[ \frac{\omega}{2}(RC + LG) \right]^2 = 0 \rightarrow c = -\left[ \frac{\omega}{2}(RC + LG) \right]^2$$

$$x = \alpha^2$$

The solution is

$$\alpha^2 = \frac{-(\omega^2 LC - RG) \pm \sqrt{(\omega^2 LC - RG)^2 + 4 \left[ \frac{\omega}{2}(RC + LG) \right]^2}}{2}$$

$$= \frac{RG - \omega^2 LC \pm \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}$$

Transmission Lines

Slide 28

## Derivation $\alpha$ and $\beta$ (5 of 7)



Step 5 – Resolve the sign of the square-root.

$$\alpha^2 = \frac{RG - \omega^2 LC \pm \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}$$

In order for this expression to always give a real value for  $\alpha$ , the sign of the square-root must be positive.

The final expression is

$$\alpha^2 = \frac{RG - \omega^2 LC + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}$$

Transmission Lines

Slide 29

## Derivation $\alpha$ and $\beta$ (6 of 7)



Step 6 – Solve for  $\beta^2$  using our expression for  $\alpha^2$ .

Recall Eq. (1b):  $\alpha^2 - \beta^2 = RG - \omega^2 LC$

$$\alpha^2 = \frac{RG - \omega^2 LC + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}$$

We obtain an equation for  $\beta^2$  by substituting our expression for  $\alpha^2$  into Eq. (1b).

$$\frac{RG - \omega^2 LC + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2} - \beta^2 = RG - \omega^2 LC$$

↓

$$\beta^2 = -\frac{RG - \omega^2 LC - \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}$$

Transmission Lines

Slide 30

## Derivation $\alpha$ and $\beta$ (7 of 7)



Step 7 – We arrive at our final expressions for  $\alpha$  and  $\beta$  in terms of the fundamental parameters  $R$ ,  $L$ ,  $G$ , and  $C$  by taking the square-root of our latest expressions for  $\alpha^2$  and  $\beta^2$ .

$$\alpha = \pm \sqrt{\frac{(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

$$\beta = \pm \sqrt{\frac{-(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

Both  $\alpha$  and  $\beta$  must be positive quantities for passive materials.  
This means we take the positive sign for the square-root.

$$\alpha = \sqrt{\frac{(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

$$\beta = \sqrt{\frac{-(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

Transmission Lines

Slide 31

# Transmission Line Parameters:

Characteristic Impedance,  $Z_0$

Transmission Lines

Slide 32



## Characteristic Impedance, $Z_0$ ( $\Omega$ )



The characteristic impedance  $Z_0$  of a transmission line is defined as the ratio of the voltage to the current at any point of a forward travelling wave.

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

Definition for a forward travelling wave.

Definition for a backward travelling wave. Notice the negative sign!

Most characteristic impedance values fall in the  $50 \Omega$  to  $100 \Omega$  range. The specific value of impedance is not usually of importance. What is important is when the impedance changes because this causes reflections, standing waves, and more.

Transmission Lines

Slide 33

## Derivation of $Z_0$ (1 of 5)



Step 1 – Substitute our solution into the transmission line equations.

$$\begin{array}{l}
 V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \\
 I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \\
 \begin{array}{cc}
 \swarrow & \searrow \\
 -\frac{dV(z)}{dz} = (R + j\omega L)I(z) & -\frac{dI(z)}{dz} = (G + j\omega C)V(z) \\
 \Downarrow & \Downarrow \\
 -\frac{d}{dz}(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}) & -\frac{d}{dz}(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}) \\
 = (R + j\omega L)(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}) & = (G + j\omega C)(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z})
 \end{array}
 \end{array}$$

Transmission Lines

Slide 34

## Derivation of $Z_0$ (2 of 5)



Step 2 – Expand the equations and calculate the derivatives.

$$-\frac{d}{dz}(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}) \\ = (R + j\omega L)(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z})$$



$$\gamma V_0^+ e^{-\gamma z} - \gamma V_0^- e^{\gamma z} \\ = (R + j\omega L)I_0^+ e^{-\gamma z} + (R + j\omega L)I_0^- e^{\gamma z}$$

$$-\frac{d}{dz}(I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}) \\ = (G + j\omega C)(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z})$$



$$\gamma I_0^+ e^{-\gamma z} - \gamma I_0^- e^{\gamma z} \\ = (G + j\omega C)V_0^+ e^{-\gamma z} + (G + j\omega C)V_0^- e^{\gamma z}$$

Transmission Lines

Slide 35

## Derivation of $Z_0$ (3 of 5)



Step 3 – Equate the expressions multiplying the common exponential terms.

$$\gamma V_0^+ = (R + j\omega L)I_0^+ \\ \gamma V_0^+ e^{-\gamma z} - \gamma V_0^- e^{\gamma z} = \underbrace{(R + j\omega L)I_0^+}_{\gamma V_0^+} e^{-\gamma z} + \underbrace{(R + j\omega L)I_0^-}_{-\gamma V_0^-} e^{\gamma z} \\ -\gamma V_0^- = (R + j\omega L)I_0^-$$

$$\gamma I_0^+ = (G + j\omega C)V_0^+ \\ \gamma I_0^+ e^{-\gamma z} - \gamma I_0^- e^{\gamma z} = \underbrace{(G + j\omega C)V_0^+}_{\gamma I_0^+} e^{-\gamma z} + \underbrace{(G + j\omega C)V_0^-}_{-\gamma I_0^-} e^{\gamma z} \\ -\gamma I_0^- = (G + j\omega C)V_0^-$$

Transmission Lines

Slide 36

## Derivation of $Z_0$ (4 of 5)



Step 4 – Solve each of our four equations for  $V_0/I_0$  to derive expressions for  $Z_0$ .

$$\gamma V_0^+ = (R + j\omega L) I_0^+ \rightarrow \frac{V_0^+}{I_0^+} = \frac{R + j\omega L}{\gamma} = Z_0$$

$$-\gamma V_0^- = (R + j\omega L) I_0^- \rightarrow -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma} = Z_0$$

$$\gamma I_0^+ = (G + j\omega C) V_0^+ \rightarrow \frac{V_0^+}{I_0^+} = \frac{\gamma}{G + j\omega C} = Z_0$$

$$-\gamma I_0^- = (G + j\omega C) V_0^- \rightarrow -\frac{V_0^-}{I_0^-} = \frac{\gamma}{G + j\omega C} = Z_0$$

Transmission Lines

Slide 37

## Derivation of $Z_0$ (5 of 5)



Step 5 – Put  $Z_0$  in terms of just  $R$ ,  $L$ ,  $G$ , and  $C$ .

Recall our expression for  $\gamma$ :  $\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$

We can substitute this into either of our expressions for  $Z_0$ .

$$Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C}$$

Proceed with the first expression.

$$Z_0 = \frac{R + j\omega L}{\sqrt{(G + j\omega C)(R + j\omega L)}} = \sqrt{\frac{(R + j\omega L)^2}{(G + j\omega C)(R + j\omega L)}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Transmission Lines

Slide 38

## Final Expression for $Z_0$ ( $\Omega$ )



We have derived a general expression for the characteristic impedance  $Z_0$  of a transmission line in terms of the fundamental parameters  $R$ ,  $L$ ,  $G$ , and  $C$ .

$$\text{Definition: } Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$$

$$\text{Expression: } Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Transmission Lines

Slide 39

## Dissecting the Characteristic Impedance, $Z_0$



The characteristic impedance describes the amplitude and phase relation between voltage and current along a transmission line. With this picture in mind, the characteristic impedance can be written as

$$Z_0 = |Z_0| \angle \theta_{Z_0}$$

$$V(z) = V_0^+ e^{-\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} = \frac{V_0^+}{Z_0} e^{-\gamma z} = |Z_0| V_0^+ e^{-\gamma z} e^{j\angle \theta_{Z_0}}$$

The characteristic impedance can also be written in terms of its real and imaginary parts.

$$Z_0 = R_0 + jX_0$$

Reactive part of  $Z_0$ . This is not equal to  $j\omega L$  or  $1/j\omega C$ .

Resistive part of  $Z_0$ . This is not equal to  $R$  or  $G$ .

Transmission Lines

Slide 40

# Special Cases of Transmission Lines:

## General Transmission Line

Transmission Lines

Slide 41

### Parameters for General TLs



Propagation Constant,  $\gamma$

$$\gamma = \alpha + j\beta = \sqrt{(G + j\omega C)(R + j\omega L)}$$

Attenuation Coefficient,  $\alpha$

$$\alpha = \sqrt{\frac{(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

Phase Constant,  $\beta$

$$\beta = \sqrt{\frac{-(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

Characteristic Impedance,  $Z_0$

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Transmission Lines

Slide 42

# Special Cases of Transmission Lines:

## Lossless Lines

Transmission Lines

Slide 43

### Definition of Lossless TL



When we think about transmission lines, we tend to think of the special case of the lossless line because the equations simplify considerably.

For a transmission line to be lossless, it must have

$$R = G = 0$$

Transmission Lines

Slide 44

## Parameters for Lossless TLs



Propagation Constant,  $\gamma$

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$$

Attenuation Coefficient,  $\alpha$

$$\alpha = 0$$

Phase Constant,  $\beta$

$$\beta = \omega\sqrt{LC}$$

Characteristic Impedance,  $Z_0$

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}$$

$$R_0 = \sqrt{\frac{L}{C}} \quad X_0 = 0$$

Transmission Lines

Slide 45

## Special Cases of Transmission Lines:

### Weakly Absorbing Line

Transmission Lines

Slide 46

## Definition of Weakly Absorbing TL



Most practical transmission lines have loss, but very low loss making them weakly absorbing.

We will define a weakly absorbing line as

$$R \leq \omega L \quad \text{and} \quad G \ll \omega C$$

Ensures low ohmic loss for signals propagating through the line.

Ensures very little conduction between the lines through the dielectric.

Transmission Lines

Slide 47

## Parameters for Weakly Absorbing TLs



Attenuation Coefficient,  $\alpha$

$$\alpha \approx \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right)$$

Conductance through the dielectric dominates attenuation in high-impedance transmission lines.

Resistivity in the conductors dominates attenuation in low-impedance transmission lines.

In weakly absorbing transmission lines, there usually exists a "sweep spot" for the impedance where attenuation is minimized.

Transmission Lines

Slide 48



# Special Cases of Transmission Lines:

## Distortionless Lines

Transmission Lines

Slide 49

### Definition of Distortionless TL



In a real transmission line, different frequencies will be attenuated differently because  $\alpha$  is a function of  $\omega$ . This causes distortion in the signals carried by the line.

$$\alpha(\omega) = \sqrt{\frac{(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}}{2}}$$

To be distortionless, there must be a choice of  $R$ ,  $L$ ,  $G$ , and  $C$  that eliminates  $\omega$  from the expression of  $\alpha$ , effectively making  $\alpha$  independent of frequency  $\omega$ .

The necessary condition to be distortionless is

$$\frac{R}{L} = \frac{G}{C}$$

Transmission Lines

Slide 50

## Parameters for Distortionless TLs



Propagation Constant,  $\gamma$

$$\gamma = \alpha + j\beta = \sqrt{RG} + j\omega\sqrt{LC}$$

Attenuation Coefficient,  $\alpha$

$$\alpha = \sqrt{RG}$$

Phase Constant,  $\beta$

$$\beta = \omega\sqrt{LC}$$

To be distortionless, we must have  $\beta \propto \omega$ .  $\beta$  is a measure of how quickly a signal accumulates phase. Different frequencies have different wavelengths and therefore must accumulate different phase through the same length of line.

Characteristic Impedance,  $Z_0$

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$$

$$R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} \quad X_0 = 0$$

Transmission Lines

Slide 51

# Example:

## Properties of RG-59 Coax

Transmission Lines

Slide 52

## The Lossless Circular Coax



Fundamental Parameters (derived in EE 3321)

$$C = \frac{2\pi\epsilon}{\ln(b/a)} \quad (\text{F/m})$$

$$L = \frac{\mu}{2\pi} \left[ \frac{1}{4} + \ln\left(\frac{b}{a}\right) \right] \quad (\text{H/m})$$

Attenuation Coefficient,  $\alpha$

$$\alpha = 0$$

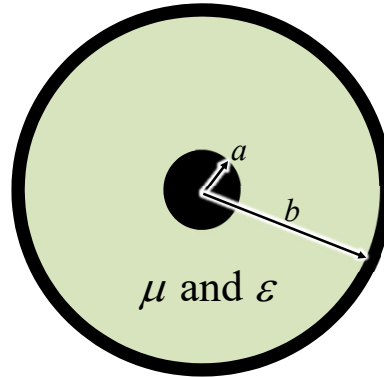
Phase Constant,  $\beta$

$$\beta = \omega\sqrt{\mu\epsilon}$$

Characteristic Impedance,  $Z_0$

$$Z_0 = R_0 + jX_0 = \frac{\eta}{2\pi} \ln\left(\frac{b}{a}\right) \quad a \ll b$$

$$R_0 = \frac{\eta}{2\pi} \ln\left(\frac{b}{a}\right) \quad X_0 = 0$$



Transmission Lines

Slide 53

## Typical RLGC for RG-59 Coax at 2 GHz



The typical RG-59 coaxial cable operating at 2.0 GHz has the following RLGC parameters:

$$R = 36 \text{ m}\Omega/\text{m}$$

$$L = 430 \text{ nH/m}$$

$$G = 10 \text{ }\mu\text{S/m}$$

$$C = 69 \text{ pF/m}$$

Calculate the transmission line parameters  $\gamma$ ,  $\alpha$ ,  $\beta$ , and  $Z_0$ .

Classify the line as lossless, weakly absorbing, distortionless, etc.

Transmission Lines

Slide 54

## Solution (1 of 3)



Our equations mostly utilize the angular frequency  $\omega$  instead of the ordinary frequency  $f$ .

$$\omega = 2\pi f = 2\pi(2.0 \times 10^9 \text{ s}^{-1}) = \underline{12.5664 \times 10^9 \text{ rad/s}}$$

The characteristic impedance  $Z_0$  is

$$\begin{aligned} Z_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\ &= \sqrt{\frac{(36 \text{ m}\Omega/\text{m}) + j(12.5664 \times 10^9 \text{ rad/s})(430 \text{ nH/m})}{(10 \text{ }\mu\text{S}/\text{m}) + j(12.5664 \times 10^9 \text{ rad/s})(69 \text{ pF/m})}} \\ &= \boxed{78.94 + j1.92 \times 10^{-4} \Omega} \end{aligned}$$

Note the imaginary part of  $Z_0$  is very small indicating that our line is very low loss.

Transmission Lines

Slide 55

## Solution (2 of 3)



The complex propagation constant  $\gamma$  is

$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{\left[ (36 \text{ m}\Omega/\text{m}) + j(12.5664 \times 10^9 \text{ rad/s})(430 \text{ nH/m}) \right]} \\ &= \sqrt{\left[ (10 \text{ }\mu\text{S}/\text{m}) + j(12.5664 \times 10^9 \text{ rad/s})(69 \text{ pF/m}) \right]} \\ &= \boxed{6.23 \times 10^{-4} + j68.45 \text{ m}^{-1}} \end{aligned}$$

From this result, we read off  $\alpha$  and  $\beta$ .

$$\gamma = \alpha + j\beta = 6.23 \times 10^{-4} + j68.45 \text{ m}^{-1}$$

$$\alpha = 6.23 \times 10^{-4} \text{ Np/m}$$

Np is Nepers

$$\beta = 68.45 \text{ rad/m}$$

rad is radians

Transmission Lines

Slide 56

## Solution (3 of 3)



Is the line lossless? → NO

No because  $R \neq 0$  and  $G \neq 0$ .  
Also, we can determine this because  $\alpha \neq 0$ .

Is the line weakly absorbing? → YES

$$R \stackrel{?}{\leq} \omega L$$

$$(36 \text{ m}\Omega/\text{m}) \stackrel{?}{\leq} (12.5664 \times 10^9 \text{ rad/s})(430 \text{ nH/m})$$

$$0.036 \stackrel{?}{\leq} 5403.5$$

Yes

$$G \stackrel{?}{\ll} \omega C$$

$$(10 \text{ }\mu\text{S/m}) \stackrel{?}{\leq} (12.5664 \times 10^9 \text{ rad/s})(69 \text{ pF/m})$$

$$10 \times 10^{-6} \stackrel{?}{\leq} 0.8671$$

Yes

Is the line distortionless? → NO, but close

$$RC \stackrel{?}{=} LG$$

$$(36 \text{ m}\Omega/\text{m})(69 \text{ pF/m}) \stackrel{?}{\leq} (430 \text{ nH/m})(10 \text{ }\mu\text{S/m})$$

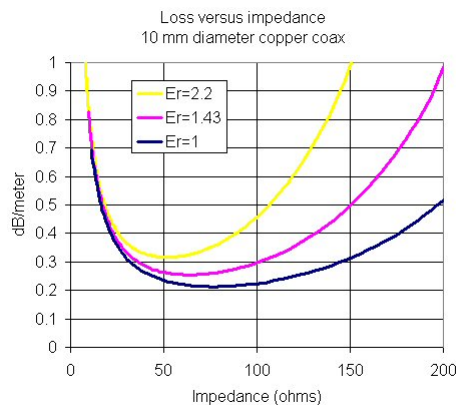
$$2.48 \times 10^{-12} \stackrel{?}{\leq} 4.30 \times 10^{-12}$$

No, but close

## Cable Loss Vs. Characteristic Impedance



As we adjust the cable dimensions (i.e.  $b/a$ ), we change both its impedance and its loss characteristics. This let's us plot the cable loss vs. characteristic impedance for a coax with different dielectric fills.



For the air-filled coax, we observe minimum loss at around  $77 \text{ }\Omega$ , where  $b/a \approx 3.5$ .

A coaxial cable filled with polyethelene ( $\epsilon_r = 2.2$ ), the minimum loss occurs at  $51.2 \text{ }\Omega$  ( $b/a = 3.6$ ).

<https://www.microwaves101.com/encyclopedias/why-fifty-ohms>

## Power Handling Vs. Characteristic Impedance

APPLIED EM

As we adjust the cable dimensions (i.e.  $b/a$ ), we affect the peak voltage handling capability (breakdown) and its power handling capability (heat).

Maximum power handling of 10 mm coax  
Voltage breakdown at 100,000 volts/meter

<https://www.microwaves101.com/encyclopedias/why-fifty-ohms>

We observe the lowest peak voltage at just over  $50 \Omega$  which we interpret as the point of best voltage handling capability.

We observe the lowest peak current at around  $30 \Omega$  which we interpret as the point of best power handling capability.

Transmission LinesSlide 59

## Why $50 \Omega$ Impedance is Best?

APPLIED EM

Two researchers, Lloyd Espenscheid and Herman Affel, working at Bell Labs produced this graph in 1929. They needed to send 4 MHz signals hundreds. Transmission lines capable of handling high voltage and high power were needed in order to accomplish this.

Data to the right was generated for an air-filled coaxial cable.

Best for High Voltage:  $Z_0 = 60 \Omega$   
 Best for High Power:  $Z_0 = 30 \Omega$   
 Best for Attenuation:  $Z_0 = 75 \Omega$

$50 \Omega$  seems like the best compromise.

Transmission Lines

Slide 60

## Why 75 $\Omega$ Impedance Standard for Coax?



Nobody really knows!!

The ideal impedance is closer to 50  $\Omega$ , however this requires a thicker center conductor. Maybe 75  $\Omega$  is a compromise between low loss and mechanical flexibility?

Transmission Lines

Slide 61


# Example:

## Microstrip Design

Transmission Lines

Slide 62

## The Lossless Microstrip



Attenuation Coefficient,  $\alpha$

$$\alpha = 0$$

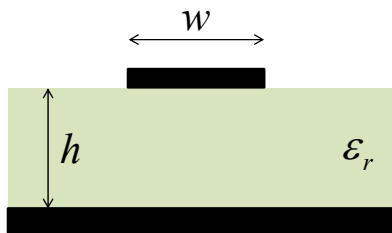
Phase Constant,  $\beta$

$$\epsilon_{r,\text{eff}} = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2\sqrt{1 + 12h/w}}$$

$$\beta \approx k_0 \sqrt{\epsilon_{r,\text{eff}}}$$


Characteristic Impedance,  $Z_0$

$$Z_0 = R_0 + jX_0 \cong \begin{cases} \frac{60}{\sqrt{\epsilon_{\text{eff}}}} \ln\left(\frac{8h}{w} + \frac{w}{4h}\right) & w/h \leq 1 \text{ thin lines} \\ \frac{1}{\sqrt{\epsilon_{\text{eff}}}} \frac{120\pi}{\left[w/h + 1.393 + 0.667 \ln(w/h + 1.444)\right]} & w/h > 1 \text{ wide lines} \end{cases}$$



Transmission Lines
Slide 63

## Problem Description



Typically, the manufacturing process fixes the value of dielectric constant  $\epsilon_r$ . This means the impedance of microstrips is controlled solely through the ratio  $w/h$ .

For this example, design a  $50 \Omega$  microstrip transmission line in FR-4, which has a dielectric constant of 4.5, to operate at 2.4 GHz.

$$\frac{w}{h} = ?$$

Transmission Lines
Slide 64



## Design Equations



To solve this problem, we must first derive some design equations. To do this, we solve our microstrip equations for  $w/h$ . This gives

$$A = \frac{Z_0}{60} \sqrt{\frac{\epsilon_r + 1}{2}} + \frac{\epsilon_r - 1}{\epsilon_r + 1} \left( 0.23 + \frac{0.11}{\epsilon_r} \right)$$

$$B = \frac{60\pi^2}{Z_0 \sqrt{\epsilon_r}}$$

$$\frac{w}{h} = \begin{cases} \frac{8e^A}{e^{2A} - 2} & w/h \leq 2 \text{ thin lines} \\ \frac{2}{\pi} \left\{ B - 1 - \ln(2B - 1) + \frac{\epsilon_r - 1}{2\epsilon_r} \left[ \ln(B - 1) + 0.39 - \frac{0.61}{\epsilon_r} \right] \right\} & w/h > 2 \text{ wide lines} \end{cases}$$

Transmission Lines

Slide 65

## Design Solution (1 of 2)



Applying our design equations, we get

$$A = 1.5438$$

$$B = 5.5831$$

$$\frac{w}{h} = \begin{cases} 1.8799 & w/h \leq 2 \text{ thin lines} \\ 1.8812 & w/h > 2 \text{ wide lines} \end{cases}$$

Since the above numbers for  $w/h$  are essentially the same, we conclude that

$$\frac{w}{h} \cong 1.88$$

Transmission Lines

Slide 66

## Design Solution (2 of 2)



We learn from our manufacturing engineer that a convenient choice for substrate thickness  $h$  is 0.5 mm. From this, to get  $50 \Omega$  the width  $w$  of the microstrip should be

$$w \cong 1.88h = 1.88(0.5 \text{ mm}) = \boxed{0.94 \text{ mm}}$$

The phase constant for this line will be

$$\epsilon_{\text{eff}} = 3.3941$$

$$k_0 = \frac{\omega}{c_0} = \frac{2\pi f}{c_0} = \frac{2\pi(2.4 \times 10^9 \text{ s}^{-1})}{299792458 \text{ m/s}} = 50.3 \text{ m}^{-1}$$

$$\beta \cong (50.3 \text{ m}^{-1})\sqrt{3.3941} = \boxed{92.67 \text{ m}^{-1}}$$