EE 4347  
Applied Electromagnetics

Topic 4d

Scattering on a Transmission Line

Lecture Outline

- Scattering at an Impedance Discontinuity
- Power on a Transmission Line
- Voltage Standing Wave Ratio (VSWR)
Scattering at an Impedance Discontinuity

Problem Setup

Transmission Line 1
\( \gamma_1, Z_1 \)

Transmission Line 2
\( \gamma_2, Z_2 \)

We will get a reflection
Incorporate Reflected Wave

Transmission Line 1

$\gamma_1, Z_1$

$z = 0$

Transmission Line 2

$\gamma_2, Z_2$

$V_1(z) = V_1^+ e^{\gamma_1 z} + V_1^- e^{\gamma_1 z}$

$I_1(z) = \frac{V_1^+}{Z_1} e^{-\gamma_1 z} - \frac{V_1^-}{Z_1} e^{\gamma_1 z}$

$V_2(z) = V_2^+ e^{-\gamma_2 z}$

$I_2(z) = \frac{V_2^+}{Z_2} e^{\gamma_2 z}$

Enforce Boundary Conditions (1 of 2)

Transmission Line 1

$\gamma_1, Z_1$

$z = 0$

Transmission Line 2

$\gamma_2, Z_2$

$V_1(z) = V_2(z)$

$V_1^+ e^{\gamma_1 z} + V_1^- e^{\gamma_1 z} = V_2^+ e^{-\gamma_2 z}$

$I_1(z) = I_2(z)$

$\frac{V_1^+}{Z_1} e^{-\gamma_1 z} - \frac{V_1^-}{Z_1} e^{\gamma_1 z} = \frac{V_2^+}{Z_2} e^{-\gamma_2 z}$

Boundary conditions require the voltage and current on either side of the interface to be equal.
Enforce Boundary Conditions (2 of 2)

Transmission Line 1

\[ V'_1(0) = V'_2(0) \]

Transmission Line 2

\[ V'^{-} + V'^{+} = V'^{-}_2 \]

The interface occurs at \( z = 0 \) so the exponential terms all equal 1.

\[ I'_1(0) = I'_2(0) \]

\[ \frac{V'^{-}_1}{Z_1} - \frac{V'^{-}_2}{Z_2} = \frac{V'^{-}_2}{Z_2} \]

Reflection Coefficient, \( \Gamma \)

Enforcing the boundary conditions at \( z = 0 \) gave us

\[ V'^{-} + V'^{+} = V'^{-}_2 \quad \text{Eq. (1)} \]

\[ \frac{V'^{-}_1}{Z_1} - \frac{V'^{-}_2}{Z_2} = \frac{V'^{-}_2}{Z_2} \quad \text{Eq. (2)} \]

Substitute Eq. (1) into Eq. (2) to eliminate \( V'^{-}_2 \).

\[ \frac{V'^{-}_1}{Z_1} - \frac{V'^{+}_1}{Z_1} = \frac{V'^{-}_1}{Z_1} + \frac{V'^{+}_1}{Z_1} \]

Solve this new expression for \( \frac{V'^{-}_1}{V'^{+}_1} \).

\[ \Gamma = \frac{\frac{V'^{-}_1}{Z_1} - \frac{Z_2}{Z_1}}{\frac{V'^{+}_1}{Z_1} - \frac{Z_2}{Z_1}} \]

\[ \Gamma = \frac{V'^{-}_1}{V'^{+}_1} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \]
Revised Equations for $V(z)$ and $I(z)$

The total voltage and current in any section of line was written as

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

Using the concept of the reflection coefficient $\Gamma$, these equations can now be written as

$$V(z) = V_0^+ e^{-\gamma z} + \Gamma V_0^- e^{\gamma z} = V_0^+ \left(e^{-\gamma z} + \Gamma e^{\gamma z}\right)$$
$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{\Gamma V_0^-}{Z_0} e^{\gamma z} = \frac{V_0^+}{Z_0} \left(e^{-\gamma z} - \Gamma e^{\gamma z}\right)$$

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \text{Reflection coefficient at the load}$$

Power on a Transmission Line
The RMS power flowing at a distance $z$ from the load is

$$P_{avg}(z) = \frac{1}{2} \text{Re}[V(z)I^*(z)]$$

* is complex conjugate

This equation is valid for any line, even those with loss.

For lossless lines (not lossless loads), we have

$$V(z) = V_0^+ \left( e^{-j\beta z} + \Gamma_L e^{j\beta z} \right) \quad I(z) = \frac{V_0^+}{Z_0} \left( e^{-j\beta z} - \Gamma_L e^{j\beta z} \right)$$

Substituting these equations into our expression for $P_{avg}(z)$ gives

$$P_{avg}(z) = \frac{1}{2} \text{Re} \left[ V_0^+ \left( e^{-j\beta z} + \Gamma_L e^{j\beta z} \right) \frac{(V_0^+)^*}{Z_0} \left( e^{-j\beta z} - \Gamma_L e^{j\beta z} \right)^* \right]$$

$$P_{avg} = \frac{|V_0|^2}{2Z_0} \left( 1 - |\Gamma_L|^2 \right)$$

Notice that the $z$ dependence vanished. This is because power flows uniformly without decay in lossless lines.

Voltage Standing Wave Ratio (VSWR)
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The VSWR is essentially the same concept as the standing wave ratio (SWR) discussed along with waves. The only difference is that it describes voltage and current instead of electromagnetic fields.

\[
\text{VSWR} = \frac{\max |V(z)|}{\min |V(z)|} = \frac{\max |I(z)|}{\min |I(z)|}
\]

Derivation of VSWR (1 of 2)

We start with our expression for waves travelling in opposite directions on a transmission line. We will assume a lossless line.

\[
V(z) = V_0^+ (e^{-j\beta z} + \Gamma_L e^{j\beta z}) \quad I(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma_L e^{j\beta z})
\]

The magnitude of the voltage signal \(V(z)\) is

\[
|V(z)| = |V_0^+| |e^{-j\beta z} + \Gamma_L e^{j\beta z}| = |V_0^+| |1 + \Gamma_L e^{2j\beta z}|
\]

By inspection of this equation, we determine the maximum and minimum values of this function.

\[
V_{\text{max}} = \max |V(z)| = |V_0^+| (1 + |\Gamma_L|) \\
V_{\text{min}} = \min |V(z)| = |V_0^+| (1 - |\Gamma_L|)
\]
The VSWR is therefore

\[ \text{VSWR} = \frac{\max V(z)}{\min V(z)} = \frac{|V_0^+| (1 + |\Gamma_L|)}{|V_0^-| (1 - |\Gamma_L|)} \rightarrow \text{VSWR} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \]

The VSWR is an easily measured quantity and we can calculate the magnitude of the reflection coefficient $|\Gamma|$ from the VSWR.

\[ |\Gamma_L| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \]

**Animation of VSWR (1 of 6)**

Case 1: 50 Ω transmission line terminated with a short-circuit load.

\[ \Gamma_L = -1 \]
Animation of VSWR (2 of 6)

Case 2: 50 Ω transmission line terminated with an open-circuit load.

\[ \Gamma_L = +1 \]

Animation of VSWR (3 of 6)

Case 3: 50 Ω transmission line terminated with a 16.5 Ω load.

\[ Z_0 > Z_L \implies \Gamma_L = -0.5 \]
Animation of VSWR (4 of 6)

Case 4: 50 $\Omega$ transmission line terminated with a 150 $\Omega$ load.

$$Z_0 < Z_L \rightarrow \Gamma_L = +0.5$$

Animation of VSWR (5 of 6)

Case 5: 50 $\Omega$ transmission line terminated with an RL load.
Case 6: 50 Ω transmission line terminated with an RC load.