EE 4347
Applied Electromagnetics

Topic 5b

Parallel Plate Waveguide

Lecture Outline

• What is a parallel plate waveguide?
• TEM Analysis
• TM Analysis
• TE Analysis
• Conclusions
What is a Parallel Plate Waveguide?

Geometry of Parallel Plate Waveguide:
Notes on the Parallel Plate Waveguide

• Becoming very popular for transmitting differential signals around a circuit board.
• Simple analysis and demonstrates most of the concepts of waveguides.
• Differential lines have confined fields for reduced interference with other devices in close proximity.
• Differential lines exhibit common mode rejection for noise immunity.

Vision for 3D High-Frequency Interconnects
TEM Analysis

Starting Point for TEM Analysis

Assuming the parallel plate waveguide has an LHI dielectric between the plates, start with the homogeneous Laplace's equation.

$$\nabla^2 V(x, y, z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

The parallel plate waveguide is uniform in the $x$ and $z$ directions so our governing equation reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Note, by assuming the field is uniform in the $x$ direction, the fringing fields at the edges are ignored.

$$\frac{\partial^2 V}{\partial y^2} = 0$$

The derivative becomes ordinary because $y$ is the only independent variable left.

$$\frac{d^2 V}{dy^2} = 0$$
How to Interpret Governing Equation

Our governing equation is now

$$\frac{d^2V}{dy^2} = 0$$

The solution to this will give $V(y)$.

$$V(y) \quad 0 \leq y \leq d$$

Boundary Conditions

Boundary conditions are needed to solve the differential equation.

$V(0) = ?$

$V(d) = ?$

Apply a voltage $V_0$ across the plates and the boundary conditions will be

$V(0) = 0$

$V(d) = V_0$
General Solution to Differential Equation

The differential equation with boundary conditions is

\[
\frac{d^2V}{dy^2} = 0 \quad 0 \leq y \leq d \quad V(0) = 0 \quad \text{and} \quad V(d) = V_0
\]

This is solved after integrating by \( y \) twice.

\[
\frac{d^2V}{dy^2} = 0
\]

\[
\frac{dV}{dy} = A
\]

\[
V(y) = Ay + B
\]

Apply Boundary Conditions

The general solution is now

\[
V(y) = Ay + B
\]

Apply the boundary condition at \( y = 0 \).

\[
V(0) = 0
\]

\[
A \cdot 0 + B = 0
\]

\[
B = 0
\]

Apply the boundary condition at \( y = d \).

\[
V(d) = V_0
\]

\[
A \cdot d + B = V_0
\]

\[
A \cdot d = V_0
\]

\[
A = \frac{V_0}{d}
\]
The Solution (1 of 2)

The final solution to the governing equation is

$$V(y) = \frac{V_0}{d} y$$

The analysis is still not finished because nothing was learned about the waveguide.

The electric field is calculated from the electric potential as

$$\vec{E} = -\nabla V = -\hat{a}_x \frac{\partial V}{\partial x} - \hat{a}_y \frac{\partial V}{\partial y} - \hat{a}_z \frac{\partial V}{\partial z}$$

$$\vec{E} = -\hat{a}_x \left( \frac{V_0}{d} \right) - \hat{a}_y \left( \frac{V_0}{d} \right) + \hat{a}_z \left( \frac{V_0}{d} \right)$$

$$\vec{E} = -\hat{a}_x \left( \frac{V_0}{d} \right) - \hat{a}_y \left( \frac{V_0}{d} \right) + \hat{a}_z \left( \frac{V_0}{d} \right)$$

$$\vec{E} = -\hat{a}_y \frac{V_0}{d}$$

The Solution (2 of 2)

If the fringing fields are ignored outside of the waveguide, the electric field is expressed as

$$\vec{E}(x,y) = \begin{cases} -\hat{a}_y \frac{V_0}{d} & \text{for } 0 \leq x \leq w \text{ and } 0 \leq y \leq d \\ 0 & \text{otherwise} \end{cases}$$
The Wave Solution

The TEM wave was derived by way of an electrostatic analysis. This ignores the wave nature of a TEM wave.

To account for the wave nature, a term must be incorporated that accumulates phase in the $z$ direction.

$$E(x, y, z) = \begin{cases} -\hat{a}_y \frac{V_0}{d} e^{-j\beta z} & \text{for } 0 \leq x \leq w \text{ and } 0 \leq y \leq d \\ 0 & \text{otherwise} \end{cases}$$

It follows that the magnetic field component is

$$H(x, y, z) = \frac{\hat{a}_z \times E}{\eta} = \frac{\hat{a}_z \times \left(-\hat{a}_y \frac{V_0}{d} e^{-j\beta z}\right)}{\eta} = -\left(\hat{a}_z \times \hat{a}_y\right) \frac{V_0}{\eta d} e^{-j\beta z} = \hat{a}_x \frac{V_0}{\eta d} e^{-j\beta z}$$

Impedance from Wave Solution (1 of 2)

The impedance of the TEM wave is defined as

$$Z_{\text{TEM}} = \frac{V_0}{I}$$

The current term $I$ must be determined. Recall the magnetic field above an infinite current sheet is

$$\vec{H}_{1 \text{ sheet}} = \frac{K \times \hat{n}}{2} \quad K = \text{surface current density (A/m)} \quad \hat{n} = -\hat{a}_z$$

Using this equation for the parallel plate waveguide ignores fringing fields at the edges.

It follows that the field between two current sheets (i.e. in our parallel plate waveguide) is

$$\vec{H}_{2 \text{ sheets}} = \vec{K} \times \hat{n}$$

Solving this for the surface current term yields

$$\vec{K} = \hat{n} \times \vec{H} = (-\hat{a}_y) \times \vec{H} = \vec{H} \times \hat{a}_y$$
Impedance from Wave Solution (2 of 2)

Find the total current $I$ by integrating the surface current across the plate.

$$I = \int_0^w (\mathbf{K} \cdot \mathbf{a}_z) \, dx = \int_0^w (\mathbf{H} \times \mathbf{a}_y) \cdot \mathbf{a}_z \, dx = \int_0^w H_x \, dx$$

Let $z = 0$ and our magnetic field solution reduces to

$$H_x (z = 0) = \frac{V_0}{\eta d}$$

Substituting this into the equation for $I$ yields

$$I = \int_0^w \left( \frac{V_0}{\eta d} \right) \, dx = \frac{V_0}{\eta d} \int_0^w \, dx = \frac{V_0}{\eta d} \cdot w = \frac{w \cdot V_0}{d \cdot \eta}$$

The characteristic impedance is found by substituting this into the original definition.

$$Z_{TEM} = \frac{V_0}{I} = \frac{w \cdot V_0}{d \cdot \eta} \quad \text{[} Z_{TEM} = \frac{\eta \cdot d}{w} \text{]}$$

Propagation Constant $\beta$

The phase constant $\beta$ cannot be calculated from the solution because it was analyzed using an electrostatic approximation.

TEM waves propagate with nearly the same speed as a plane wave in an infinite medium composed of the dielectric that is between the plates.

$$\beta_{TEM} \approx \omega \sqrt{\mu \varepsilon}$$
Distributed Inductance and Capacitance

The characteristic impedance of the parallel plate transmission line is

\[ Z_{\text{TEM}} = \eta \frac{d}{w} \]

The distributed capacitance \( C \) can be estimated as

\[ C = \varepsilon \frac{w}{d} \]

It follows that the distributed inductance \( L \) is

\[ Z_{\text{TEM}} = \eta \frac{d}{w} = \sqrt{\frac{L}{C}} = \sqrt{\frac{L}{\varepsilon (w/d)}} \quad L = \mu \frac{d}{w} \]

Example #1 (1 of 3)

Given the following parallel plate waveguide...

\[ w = 2.0 \text{ mm} \quad d = 0.5 \text{ mm} \quad \varepsilon_r = 2.3 \]

What is the characteristic impedance \( Z_0 \)?

What value of \( w \) would make this transmission line 50 \( \Omega \)?
Example #1 (2 of 3)

The equation for characteristic impedance is

$$ Z_{TEM} = \eta \frac{d}{w} $$

The impedance \( \eta \) of the dielectric is

$$ \eta = \eta_0 \sqrt{\frac{\mu_r}{\varepsilon_r}} = \left(376.73 \ \Omega\right) \sqrt{\frac{1.0}{2.3}} = 248.4 \ \Omega $$

The characteristic impedance is therefore

$$ Z_{TEM} = \frac{\eta d}{w} = \frac{(248.4 \ \Omega)(0.5 \ \text{mm})}{(2.0 \ \text{mm})} = 62.1 \ \Omega $$

Example #1 (3 of 3)

Solve the equation for characteristic impedance for \( w \).

$$ Z_{TEM} = \eta \frac{d}{w} \quad \Rightarrow \quad w = \frac{\eta d}{Z_{TEM}} $$

To get 50 \( \Omega \), \( w \) must be

$$ w = \frac{\eta d}{Z_0} = \frac{(248.4 \ \Omega)(0.5 \ \text{mm})}{(50 \ \Omega)} = 2.48 \ \text{mm} $$
### Visualization of TEM Mode

- Field Solution
- $E_x (x, y, z) = 0$
- $E_y (x, y, z) = \frac{V_0}{d} e^{-j\beta z}$
- $E_z (x, y, z) = 0$
- $H_y (x, y, z) = \frac{V_0}{\eta d} e^{-j\beta z}$
- $H_z (x, y, z) = 0$
- $H_x (x, y, z) = 0$

### Summary of TEM Analysis

- Field Solution
  - $E_x (x, y, z) = 0$
  - $E_y (x, y, z) = \frac{V_0}{d} e^{-j\beta z}$
  - $E_z (x, y, z) = 0$
  - $H_y (x, y, z) = \frac{V_0}{\eta d} e^{-j\beta z}$
  - $H_z (x, y, z) = 0$
  - $H_x (x, y, z) = 0$

- Phase Constant
  - $\beta_{\text{TEM}} = \omega \sqrt{\mu \varepsilon}$
  - Same as plane wave.

- Cutoff Frequency
  - $f_c = 0$
  - No cutoff frequency.
  - Mode supported at DC.

- Characteristic Impedance
  - $Z_{\text{TEM}} = \eta \frac{d}{w}$

- TEM has no cutoff frequency
- TEM is the TM_0 mode.
Recall the Starting Point

The governing equation for TE analysis is

\[
\frac{\partial^2 H_{0,z}}{\partial x^2} + \frac{\partial^2 H_{0,z}}{\partial y^2} - k_e^2 H_{0,z} = 0 \quad k_e^2 = k^2 - \beta^2
\]

After a solution is obtained, the remaining field components are calculated according to

\[
H_{0,x} = -\frac{j\beta}{k_e} \frac{\partial H_{0,z}}{\partial x} \quad E_{0,x} = -\frac{j\omega\mu}{k_e} \frac{\partial H_{0,z}}{\partial y}
\]

\[
H_{0,y} = -\frac{j\beta}{k_e} \frac{\partial H_{0,z}}{\partial y} \quad E_{0,y} = \frac{j\omega\mu}{k_e} \frac{\partial H_{0,z}}{\partial x}
\]

\[
E_{0,z} = 0
\]
**Simplify Governing Equation**

Assuming the waveguide is uniform in the direction of $x$

\[
\frac{\partial}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2} = 0
\]

The governing equation reduces to

\[
\frac{\partial^2 H_{0,z}}{\partial x^2} + \frac{\partial^2 H_{0,y,z}}{\partial y^2} - k_z^2 H_{0,z} = 0 \quad \rightarrow \quad \frac{d^2 H_{0,z}}{dy^2} - k_z^2 H_{0,z} = 0
\]

**General Solution**

The general solution to the governing equation is

\[
\frac{d^2 H_{0,z}}{dy^2} - k_z^2 H_{0,z} = 0 \quad \rightarrow \quad H_{0,z} = A \sin(k_z y) + B \cos(k_z y)
\]
Boundary Conditions (1 of 2)

The electric field must be zero at the plates.
The solution, however, is in terms of the magnetic field.
The solution must be written in terms of an electric field.
The only component of the electric field tangential to the interface is $E_{0,x}$:

$$E_{0,x} = -rac{j \omega \mu}{k_c^2} \frac{\partial H_{0,z}}{\partial y} = -rac{j \omega \mu}{k_c^2} \frac{\partial}{\partial y} \left[ A \sin(k_c y) + B \cos(k_c y) \right]$$

$$= -rac{j \omega \mu}{k_c} \left[ A \cos(k_c y) - B \sin(k_c y) \right]$$

Boundary Conditions (2 of 2)

The first boundary condition is

$$E_{0,x}(x,0) = 0$$

$$E_{0,x}(x,0) = -rac{j \omega \mu}{k_c} \left[ A \cos(0) - B \sin(0) \right] = -rac{j \omega \mu}{k_c} A = 0 \quad \rightarrow \quad A = 0$$

The second boundary condition is

$$E_{0,x}(x,d) = 0$$

$$E_{0,x}(x,d) = -rac{j \omega \mu}{k_c} \left[ -B \sin(k_c d) \right] = B \frac{j \omega \mu}{k_c} \sin(k_c d)$$

$B = 0$ cannot be chosen because that would lead to a trivial solution. Instead, it must be the sine term that is zero at $y = d$.

$$\sin(k_c d) = 0 \quad \rightarrow \quad k_c d = m \pi \quad n = 1, 2, 3, ...$$

The cutoff wave number is then

$$k_c = \frac{m \pi}{d} \quad m = 1, 2, 3, ...$$

Note that $m = 0$ would force the entire field to be zero so this is not a valid solution.
Recall the definition of the cutoff wave number. Solve this for $\beta$.

\[ k_c^2 = k^2 - \beta^2 \quad \rightarrow \quad \beta = \sqrt{k^2 - k_c^2} \]

An expression for $k_c$ was previously derived that arose from the boundary conditions.

\[ \beta = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad m = 1, 2, 3, \ldots \]

From this, it is observed that there are an infinite number of discrete solutions and the order of the mode is $m$.

\[ \beta_m = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad m = 1, 2, 3, \ldots \]
Final Solution

Recall the general solution was
\[ \frac{d^2 H_{0,z}}{dy^2} - k_c^2 H_{0,z} = 0 \quad \Rightarrow \quad H_{0,z} = A \sin(k_c y) + B \cos(k_c y) \]

But now it is known that \( A = 0 \) and \( k_c = \frac{m \pi}{d} \). The final solution is
\[ H_{0,z}(x, y) = B_m \cos\left(\frac{m \pi y}{d}\right) \quad \Rightarrow \quad H_z(x, y, z) = B_m \cos\left(\frac{m \pi y}{d}\right) e^{-i \beta z} \]

The remaining field components are calculated from this result.

\[
\begin{align*}
H_x &= -\frac{j \beta_m}{k_c} \frac{\partial}{\partial x} \left[ B_m \cos\left(\frac{m \pi y}{d}\right) e^{-i \beta z}\right] = 0 \\
H_y &= -\frac{j \beta_m}{k_c} \frac{\partial}{\partial y} \left[ B_m \cos\left(\frac{m \pi y}{d}\right) e^{-i \beta z}\right] = j \frac{\beta_m}{k_c} B_m \sin\left(\frac{m \pi y}{d}\right) e^{-i \beta z} \\
E_z &= j \frac{\beta_m}{k_c} \frac{\partial}{\partial z} \left[ B_m \cos\left(\frac{m \pi y}{d}\right) e^{-i \beta z}\right] = 0 \\
E_x &= 0 \\
E_y &= 0
\end{align*}
\]

Why No TE_0 Mode?

For \( m = 0 \), the field components are
\[
\begin{align*}
H_x &= 0 \\
H_y &= \frac{j \beta_m}{k_c} B_m \sin(0) e^{-i \beta z} = 0 \\
H_z &= B_m \cos(0) e^{-i \beta z} = B_m e^{-i \beta z} \\
E_x &= \frac{j \omega \mu}{k_c} B_m \sin(0) e^{-i \beta z} = 0 \\
E_y &= 0 \\
E_z &= 0
\end{align*}
\]

This is not a physical solution because the electric field is completely zero.
Cutoff Condition

Recall the phase constant is calculated as

\[
\beta_m = \sqrt{k_c^2 - k^2} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad m = 1, 2, 3, \ldots
\]

This becomes imaginary when \( k_c > k \). Values of \( m \) that cause this condition correspond to modes that are “cutoff.” These are still modes, but they decay very quickly so they are almost never used.

\[
k_c = \omega \sqrt{\mu \varepsilon} = \frac{m\pi}{d}
\]

\[
2\pi f_c \sqrt{\mu \varepsilon} = \frac{m\pi}{d}
\]

\[
f_c = \frac{m}{2d \sqrt{\mu \varepsilon}} = \frac{k_c}{2\pi \sqrt{\mu \varepsilon}}
\]

This is the cutoff frequency for the TE\(_m\) mode.

Characteristic Impedance \( Z_{TE} \)

The characteristic impedance is defined as

\[
Z_{TE} = \frac{E_{0,x}}{H_{0,y}} = -\frac{E_{0,y}}{H_{0,x}}
\]

The expressions for the field quantities are substituted into this to obtain

\[
Z_{TE} = \frac{E_{0,x}}{H_{0,y}} = \frac{j\omega \mu}{k_c} B_m \sin \left(\frac{n\pi y}{d}\right) = \frac{\omega \mu}{\beta_m} = \frac{\eta k}{\beta_m}
\]
Effective Refractive Index $n_{\text{eff}}$

A wave propagates in a waveguide at a speed quantified by the effective refractive index $n_{\text{eff}}$.

$$\beta_m = k_0 n_{\text{eff}} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad \Rightarrow \quad n_{\text{eff}} = n \sqrt{1 - \left(\frac{m\lambda_0}{2nd}\right)^2}$$

This term acts to make $n_{\text{eff}} < n$.

$n$ is the refractive index of the dielectric in the waveguide.

Visualization of TE$_1$ Mode

$\vec{E}$

$\vec{H}$
Visualization of TE₂ Mode

Visualization of TE₃ Mode
Summary of TE Analysis

Field Solution

\[ E_\parallel(x, y, z) = \frac{j\omega}{k_z} B_n \sin \left( \frac{m\pi y}{d} \right) e^{-j\beta_z z} \]
\[ E_\perp(x, y, z) = 0 \]
\[ H_z(x, y, z) = 0 \]
\[ H_\parallel(x, y, z) = j\frac{\beta_n}{k_z} B_n \sin \left( \frac{m\pi y}{d} \right) e^{-j\beta_z z} \]
\[ H_\perp(x, y, z) = B_n \cos \left( \frac{m\pi y}{d} \right) e^{-j\beta_z z} \]

Phase Constant

\[ \beta_n = \sqrt{k^2 - \left( \frac{m\pi}{d} \right)^2} \]
\[ m = 1, 2, 3, \ldots \]

Same equation as for TM

Cutoff Frequency

\[ f_c = \frac{m}{2d\sqrt{\mu\epsilon}} \]

Same equation as for TM

Characteristic Impedance

\[ Z_{TE,n} = \frac{k\eta}{\beta_n} \]

- TE\_0 mode does not exist
- TE\_1 is the lowest order TE mode
Recall the Starting Point

The governing equation for TM analysis is

\[ \frac{\partial^2 E_{0,z}}{\partial x^2} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0 \]

\[ k_c^2 = k^2 - \beta^2 \]

After a solution is obtained, the remaining field components are calculated according to

\[ H_{0,z} = \frac{j \omega \varepsilon E_{0,z}}{k_c^2} \frac{\partial E_{0,z}}{\partial y} \]

\[ E_{0,z} = -\frac{j \beta \varepsilon E_{0,z}}{k_c^2} \frac{\partial E_{0,z}}{\partial x} \]

Simplify Governing Equation

Assuming the waveguide is uniform in the direction of \( x \)

\[ \frac{\partial}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial x^2} = 0 \]

The governing equation reduces to

\[ \frac{\partial E_{0,z}}{\partial x} + \frac{\partial^2 E_{0,z}}{\partial y^2} - k_c^2 E_{0,z} = 0 \]

\[ \frac{d^2 E_{0,z}}{dy^2} - k_c^2 E_{0,z} = 0 \]
**General Solution**

The general solution to the governing equation is

\[ \frac{d^2 E_{0,z}}{dy^2} - k_z^2 E_{0,z} = 0 \quad \rightarrow \quad E_{0,z} = A \sin(k_z y) + B \cos(k_z y) \]

**Boundary Conditions**

The electric field component \( E_{0,z} \) is tangential to the interfaces to the boundary conditions are applied to this directly.

The first boundary condition is

\[ E_{0,z}(x,0) = A \sin(0) + B \cos(0) = B = 0 \quad \rightarrow \quad B = 0 \]

The second boundary condition is

\[ E_{0,z}(x,d) = A \sin(k_z d) = 0 \]

\( A = 0 \) cannot be chosen because that would lead to a trivial solution. Instead, it must be sine term that is zero at \( y = d \).

\[ \sin(k_z d) = 0 \quad \rightarrow \quad k_z d = m\pi \quad m = 0,1,2,3,... \]

The cutoff wave number is then

\[ k_z = \frac{m\pi}{d} \quad m = 0,1,2,3,... \]

Note that \( m = 0 \) is allowed in this case because it does not force the field to be zero. It does, however, force the field to be uniform. Thus TM\(_0\) is the TEM mode.
Phase Constant $\beta$

Recall the definition of the cutoff wave number. Solve this for $\beta$.

$$ k_c^2 = k^2 - \beta^2 \quad \rightarrow \quad \beta = \sqrt{k^2 - k_c^2} $$

Here is an expression for $k_c$ that arose from the boundary conditions.

$$ \beta = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad m = 0, 1, 2, 3, \ldots $$

It is observed that there is an infinite number of discrete solutions where the order of the mode is $m$.

$$ \beta_m = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad m = 0, 1, 2, 3, \ldots $$

Final Solution

Recall the general solution was

$$ \frac{d^2 E_{0,z}}{dz^2} - k_c^2 E_{0,z} = 0 \quad \rightarrow \quad E_{0,z} = A \sin(k_c y) + B \cos(k_c y) $$

But now it is known that $B = 0$ and $k_c = \frac{m\pi}{d}$. The final solution is

$$ E_{0,z}(x, y) = A_m \sin\left(\frac{m\pi y}{d}\right) \quad \rightarrow \quad E_z(x, y, z) = A_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z} $$

The remaining field components are calculated from this result.

$$ H_x = \frac{j \omega \varepsilon E_y}{k_c^2} = \frac{j \omega \varepsilon}{k_c^2} \left[ A_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z} \right] = \frac{j \omega \varepsilon}{k_c} A_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z} $$

$$ H_y = \frac{j \omega \varepsilon E_z}{k_c^2} = \frac{j \omega \varepsilon}{k_c^2} \left[ A_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z} \right] = 0 $$

$$ H_z = 0 $$

$$ E_x = \frac{j \beta_e \varepsilon E_z}{k_c^2} = \frac{j \beta_e \varepsilon}{k_c^2} \left[ A_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z} \right] = 0 $$

$$ E_y = \frac{j \beta_e \varepsilon E_z}{k_c^2} = \frac{j \beta_e \varepsilon}{k_c^2} \left[ A_m \sin\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z} \right] = -\frac{j \beta_e}{k_c} A_m \cos\left(\frac{m\pi y}{d}\right) e^{-j\beta_m z} $$
**Why Does TM$_0$ Mode Exist?**

For $m = 0$, the field components are

\[ H_x = \frac{j\omega}{k_c} A_m \cos(0) e^{-j\beta_m z} = \frac{j\omega}{k_c} A_m e^{-j\beta_m z} \]
\[ H_y = 0 \]
\[ H_z = 0 \]
\[ E_x = 0 \]
\[ E_y = -\frac{j\beta_m}{k_c} A_m \cos(0) e^{-j\beta_m z} = -\frac{j\beta_m}{k_c} A_m e^{-j\beta_m z} \]
\[ E_z(x, y, z) = A_m \sin(0) e^{-j\beta_m z} = 0 \]

This is a valid solution.

**Cutoff Condition**

Recall that the phase constant calculated as

\[ \beta_m = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{d}\right)^2} \quad m = 0, 1, 2, 3, \ldots \]

This becomes imaginary when $k_c > k$. Values of $m$ that cause this condition correspond to modes that are “cutoff.” These are still modes, but they decay very quickly so they are almost never used.

\[ k_c = \omega \sqrt{\mu \varepsilon} = \frac{m\pi}{d} \]
\[ 2\pi f_c \sqrt{\mu \varepsilon} = \frac{m\pi}{d} \]
\[ f_c = \frac{m}{2d \sqrt{\mu \varepsilon}} = \frac{k_c}{2\pi \sqrt{\mu \varepsilon}} \quad \text{This is the cutoff frequency for the TM$_m$ mode.} \]
Characteristic Impedance $Z_{TM}$

The characteristic impedance is defined as

$$Z_{TM} = \frac{E_{0,x}}{H_{0,y}} = -\frac{E_{0,y}}{H_{0,x}}$$

The expressions for these field quantities are substituted in to obtain

$$Z_{TM} = -\frac{E_{0,y}}{H_{0,x}} = -\frac{j\beta_m A_m}{k_c} \cos \left( \frac{m\pi y}{d} \right) = \frac{\beta_m}{\omega \varepsilon} = \eta \frac{\beta_m}{k}$$
Visualization of TM₁ Mode

Visualization of TM₂ Mode
Example #2 (1 of 2)

Given the following parallel plate waveguide...

\[ w = 2.48 \text{ mm} \]
\[ d = 0.5 \text{ mm} \]
\[ \varepsilon_r = 2.3 \]

What is the bandwidth of this waveguide when used as a transmission line?

Example #2 (2 of 2)

When used as a transmission line, it is only the TEM mode that is of interest. The bandwidth is the range of frequencies for which the waveguide supports only the TEM mode.

The cutoff frequencies are the same for the TE and TM modes, so they are essentially checked at the same time.

The second order modes are TE₁ and TM₁. The bandwidth is simply the cutoff frequency of these modes.

\[
f_c(m = 1) = \frac{k_c}{2\pi\sqrt{\mu\varepsilon}} = \frac{mc_0}{2d\sqrt{\mu\varepsilon}} = \frac{(1)(299792458 \text{ m/s})}{2(0.5 \text{ mm})\sqrt{(1.0)(2.3)}} = 197.6 \text{ GHz}
\]
Summary of TM Analysis

Field Solution

\[ E_x(x, y, z) = 0 \]
\[ E_y(x, y, z) = -j\beta_n \frac{A_n}{k_y} \cos \left( \frac{m\pi y}{d} \right) e^{-j\beta_n z} \]
\[ E_z(x, y, z) = A_n \sin \left( \frac{m\pi y}{d} \right) e^{-j\beta_n z} \]
\[ H_x(x, y, z) = \frac{j\omega c}{k_z} A_n \cos \left( \frac{m\pi y}{d} \right) e^{-j\beta_n z} \]
\[ H_y(x, y, z) = 0 \]
\[ H_z(x, y, z) = 0 \]

Phase Constant

\[ \beta_n = \sqrt{k^2 - \left( \frac{m\pi}{d} \right)^2} \]
\[ m = 0, 1, 2, 3, \ldots \]

Same equation as for TE

Cutoff Frequency

\[ f_{c,n} = \frac{m}{2d\sqrt{\mu\varepsilon}} \]

Same equation as for TE

Characteristic Impedance

\[ Z_{TM,n} = \frac{\beta_n}{k} \]

Conclusion
Summary of Parallel Plate Waveguide

<table>
<thead>
<tr>
<th>Parameter</th>
<th>TEM ( m = 0,1,2,3\ldots )</th>
<th>TM ( m = 0,1,2,3\ldots )</th>
<th>TE ( m = 1,2,3\ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( \omega \sqrt{\mu \varepsilon} )</td>
<td>( \omega \sqrt{\mu \varepsilon} )</td>
<td>( \omega \sqrt{\mu \varepsilon} )</td>
</tr>
<tr>
<td>( k_c )</td>
<td>0</td>
<td>( m \pi / d )</td>
<td>( m \pi / d )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>( k - \omega \sqrt{\mu \varepsilon} )</td>
<td>( \sqrt{k^2 - k_c^2} )</td>
<td>( \sqrt{k^2 - k_c^2} )</td>
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<tr>
<td>( \lambda_c )</td>
<td>( \infty )</td>
<td>( 2 \pi k_c )</td>
<td>( 2 \pi k_c )</td>
</tr>
<tr>
<td>( \lambda_g )</td>
<td>( 2 \pi / k )</td>
<td>( 2 \pi / \beta_m )</td>
<td>( 2 \pi / \beta_m )</td>
</tr>
<tr>
<td>( \nu_p )</td>
<td>( \omega k - \omega \sqrt{\mu \varepsilon} )</td>
<td>( \omega / \beta_m )</td>
<td>( \omega / \beta_m )</td>
</tr>
<tr>
<td>( \alpha_d )</td>
<td>( k \tan \delta / 2 )</td>
<td>( k^\prime \tan \delta / 2 \beta_m )</td>
<td>( k^\prime \tan \delta / 2 \beta_m )</td>
</tr>
<tr>
<td>( \alpha_c )</td>
<td>( R_{c,pd} )</td>
<td>( 2 \nu_p R_{c,pd} )</td>
<td>( 2 \nu_p R_{c,pd} )</td>
</tr>
<tr>
<td>( E_x )</td>
<td>0</td>
<td>0</td>
<td>( (j \nu_p k) B_x \sin(m \pi y / d) e^{-j \omega r} )</td>
</tr>
<tr>
<td>( E_y )</td>
<td>( -Y_{c,pd} e^{-j \omega r} )</td>
<td>( -j \nu_p k B_x \cos(m \pi y / d) e^{-j \omega r} )</td>
<td>0</td>
</tr>
<tr>
<td>( E_z )</td>
<td>0</td>
<td>( \nu_p \sin(m \pi y / d) e^{j \omega r} )</td>
<td>0</td>
</tr>
<tr>
<td>( H_x )</td>
<td>( -Y_{c,pd} e^{-j \omega r} )</td>
<td>( j \nu_p k B_x \cos(m \pi y / d) e^{j \omega r} )</td>
<td>0</td>
</tr>
<tr>
<td>( H_y )</td>
<td>0</td>
<td>0</td>
<td>( B_z \cos(m \pi y / d) e^{j \omega r} )</td>
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<tr>
<td>( H_z )</td>
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<td>0</td>
<td>( j \nu_p k B_x )</td>
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<tr>
<td>( Z )</td>
<td>( \nu_p l_w )</td>
<td>( \beta_m / k )</td>
<td>( k \beta_m )</td>
</tr>
</tbody>
</table>

Modes in Parallel Plate Waveguide

m = 0

m = 1

m = 2

m = 3
Notes

- Supports TEM mode when it has a homogeneous dielectric because it has two conductors.
- Supports TE and TM modes when it has a homogeneous dielectric
- The lowest order mode is TM\textsubscript{0} which is the TEM mode.