Topic 4

Numerical Analysis of Transmission Lines

Outline

• Solution Approach
  – Do not need to understand completely

• Using tlcalc.p to Analyze Transmission Lines
  – Must understand completely

• Examples
Solution Approach

Maxwell’s Equations

We start with Maxwell’s equations.

\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \cdot \vec{D} = 0 \]
\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
Small Size Approximation

The dimensions of a transmission are typically much smaller than the operating wavelength so the wave nature of electromagnetics is less important to consider. Therefore, we are essentially solving Maxwell’s equations as $d/dt \to 0$.

\[
\nabla \cdot \vec{B} = 0 \\
\nabla \cdot \vec{D} = 0 \\
\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \\
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\]

Electrostatics & Magnetostatics

Maxwell’s equations have decoupled into two sets of equations. Once describes electrostatics while the other describes magnetostatics.

**Electrostatics**

\[
\nabla \cdot \vec{B} = 0 \\
\nabla \cdot \vec{D} = 0 \\
\nabla \times \vec{H} = \vec{J} \\
\nabla \times \vec{E} = 0
\]

**Magnetostatics**

\[
\nabla \cdot \vec{B} = 0 \\
\nabla \cdot \vec{D} = 0 \\
\nabla \times \vec{H} = \vec{J} \\
\nabla \times \vec{E} = 0
\]
Governing Equations

Maxwell’s equations:
\[ \nabla \cdot \vec{D} = 0 \quad \text{Eq. (1)} \]
\[ \nabla \times \vec{E} = 0 \quad \text{Eq. (2)} \]

In addition, we have the constitutive relation
\[ \vec{D} = \varepsilon_r \vec{E} \quad \text{Eq. (3)} \]

We do not like to solve vector equations if we do not have to. Electrostatic fields are completely characterized by the scalar potential \( V \).
\[ \vec{E} = -\nabla V \quad \text{Eq. (4)} \]

Differential Equation to Solve

1. Substitute Eq. (3) into Eq. (1) to eliminate \( D \).
\[ \nabla \cdot (\varepsilon_r \vec{E}) = 0 \quad \text{Eq. (5)} \]

2. Substitute Eq. (4) into Eq. (5) to eliminate \( E \).
\[ \nabla \cdot \left[ \varepsilon_r \left( \nabla V \right) \right] = 0 \quad \text{Eq. (6)} \]
Derive Finite-Difference Equation

When the dielectric is homogeneous, our differential equation simplifies to

$$\nabla \cdot \left[ \varepsilon_r \left( \nabla V \right) \right] = 0 \quad \rightarrow \quad \nabla^2 V = 0$$

In Cartesian coordinates, this expands to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Now, the function $V(x, y)$ is made discrete and the derivatives are approximated using finite-differences.

$$V(i+1, j) - 2V(i, j) + V(i-1, j)\frac{x}{(\Delta x)^2} + V(i, j+1) - 2V(i, j) + V(i, j-1)\frac{y}{(\Delta y)^2} = 0$$

Write Large Set of Equations

We rearrange our finite-difference equation to the following form

$$\frac{1}{(\Delta y)^2} V(i-1, j) + \frac{1}{(\Delta y)^2} V(i+1, j) + \frac{2}{(\Delta x)^2} V(i, j) + \frac{1}{(\Delta y)^2} V(i, j-1) = 0$$

This equation is written once for every point on the grid.

$N_x = 4 \quad \Delta x = 0.5$

$N_y = 4 \quad \Delta y = 0.5$
Consider Boundary Conditions

\[ N_x = 4 \quad \Delta x = 0.5 \]
\[ N_y = 4 \quad \Delta y = 0.5 \]

We will set all of the highlighted terms to zero

→ Dirichlet Boundary Conditions

Build Matrix

\[
\begin{bmatrix}
+4 \cdot V(1,1) + 16 \cdot V(1,1) & +4 \cdot V(1,2) & 0 \\
+4 \cdot V(2,1) + 4 \cdot V(3,1) + 16 \cdot V(2,1) & +4 \cdot V(2,2) & 0 \\
+4 \cdot V(2,1) + 4 \cdot V(1,1) + 16 \cdot V(2,1) & +4 \cdot V(2,2) & 0 \\
+4 \cdot V(3,1) + 4 \cdot V(1,1) + 16 \cdot V(3,1) & +4 \cdot V(3,2) & 0 \\
+4 \cdot V(2,1) + 16 \cdot V(4,1) & +4 \cdot V(4,2) & 0 \\
+4 \cdot V(2,1) + 4 \cdot V(2,2) + 16 \cdot V(1,2) + 4 \cdot V(1,1) & +4 \cdot V(1,3) & 0 \\
+4 \cdot V(1,2) + 4 \cdot V(3,2) + 16 \cdot V(2,2) + 4 \cdot V(2,1) & +4 \cdot V(2,3) & 0 \\
+4 \cdot V(2,2) & +4 \cdot V(2,3) & 0 \\
+4 \cdot V(3,2) & +4 \cdot V(3,3) & 0 \\
+4 \cdot V(1,4) & +4 \cdot V(4,4) + 16 \cdot V(3,4) & +4 \cdot V(3,3) & 0 \\
+4 \cdot V(2,4) & +4 \cdot V(4,4) + 16 \cdot V(3,4) & +4 \cdot V(3,3) & 0 \\
+4 \cdot V(3,4) & +4 \cdot V(4,4) + 16 \cdot V(4,4) & +4 \cdot V(4,3) & 0 \\
+4 \cdot V(4,4) & +4 \cdot V(4,5) & 0 \\
\end{bmatrix}
\]
Is Our Matrix Equation Solvable?

\[ \nabla^2 V = 0 \rightarrow [L][v] = [0] \rightarrow [v] = [L]^{-1}[0] = [0] \]

Trivial Solution

Force Known Potentials
Solve for Scalar Potential

\[ [L][v]=[b] \quad \rightarrow \quad [v]=[L]^{-1}[b] \]

Calculating the Fields

Once the scalar potential is solved, the electric field intensity is

\[ \vec{E}(x, y) = -\nabla V(x, y) \]

The electric flux density is then

\[ \vec{D}(x, y) = \varepsilon_r(x, y) \vec{E}(x, y) \]
Distributed Capacitance

In the electrostatic approximation, the transmission line is a capacitor. The total energy $U$ stored in a capacitor is

$$U = \frac{1}{2} \iint_A (\vec{D} \cdot \vec{E}) dA$$

The capacitance $C$ is related to the total stored energy $U$ through

$$U = \frac{CV_0^2}{2}$$

$V_0$ is the voltage across the capacitor.

If we set the above equations equal and solve for $C$, we get

$$C = \frac{1}{V_0^2} \iint_A (\vec{D} \cdot \vec{E}) dA$$

Distributed Inductance

The voltage signal along the transmission line travels at the same velocity as the electromagnetic field so we can write

$$v_f = v_E \quad \Rightarrow \quad \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_e}} \quad \Rightarrow \quad LC = \frac{\mu_e}{c_0^2}$$

Solving this for $L$ we get

$$L = \frac{\mu_e}{c_0^2}$$

This means we can calculate the distributed inductance $L$ directly from the distributed capacitance.

Dielectric materials should not alter the inductance. However if we use the value of $C$ calculated on the previous slide, it will. This is incorrect. The solution is to calculate the distributed capacitance $C_H$ with air dielectric and then calculate the distributed inductance from this.

$$L = \frac{\mu_e}{c_0^2 C_H}$$
Calculating the Transmission Line Parameters

The characteristic impedance is calculated from the distributed inductance and capacitance through

\[ Z_c = \sqrt{\frac{L}{C}} \]

The phase constant is

\[ \beta = \omega \sqrt{LC} = k_0 n_{\text{eff}} \]

Recall, both \( Z_c \) and \( \beta \) are needed to analyze transmission line circuits.

Note: It was assumed \( R = G = 0 \) (lossless transmission line)

Using `tlcalc.p` to Analyze Transmission Lines
The Short Story

\[
[Z_0, n, L, C, V, E_x, E_y] = \text{tlcalc}(ER, C1, C2, dx, dy);
\]

This MATLAB function calculates the properties of an arbitrary transmission.

- `ER` is a 2D array describing the relative permittivity.
- `C1` is a 2D array identifying the points of conductor 1.
- `C2` is a 2D array identifying the points of conductor 2.
- `dx` and `dy` are the grid resolution parameters.

To simulate the line `C1` is set to 1 volt and `C2` is set to 0 volts.

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Number Values for Inputs

Permittivity Function

\[ 1 \leq \varepsilon_r(x, y) < \infty \]

\( \varepsilon_r \) should be purely real.

Loss would be accounted for differently.

Conductor #1

0 \equiv \text{dielectric}

1 \equiv \text{metal}

Array should be all 0's except 1's in the positions that describe the first conductor.

Conductor #2

0 \equiv \text{dielectric}

1 \equiv \text{metal}

Array should be all 0's except 1's in the positions that describe the second conductor.

Grid Resolution

Pick your smallest dimension and resolve this with at least 5 to 10 points.

\[ dx = w/10; \]
\[ dy = dx; \]
Space Around Transmission Line

Add enough space between transmission line and the boundary to ensure the electric field decays sufficiently. Remember, we have used Dirichlet boundary conditions.

Thickness of Conductors

Conductors are usually very thin (20-50 micrometers).

This is not usually feasible to resolve in this simple analysis tool.

Just make the conductors one cell thick, unless you have very thick conductors.
Scalability

These TLs will have the exact same properties.
Electrostatics has no fundamental size scale.

Examples
Coplanar Transmission Line

\[ L = 317.3 \text{ nH/m} \]
\[ C = 59.9 \text{ pF/m} \]
\[ Z_0 = 72.8 \Omega \]
\[ n_{\text{eff}} = 1.3066 \]

Microstrip Transmission Line

\[ L = 304.7 \text{ nH/m} \]
\[ C = 116.8 \text{ pF/m} \]
\[ Z_0 = 51.1 \Omega \]
\[ n_{\text{eff}} = 1.7881 \]
Parallel Plate Transmission Line

$L = 263.3 \text{ nH/m} \quad C = 105.7 \text{ pF/m} \quad Z_0 = 49.9 \Omega \quad n_{\text{eff}} = 1.5811$

Coaxial Transmission Line

$L = 379.5 \text{ nH/m} \quad C = 64.5 \text{ pF/m} \quad Z_0 = 76.7 \Omega \quad n_{\text{eff}} = 1.4832$
Symmetric Stripline

$L = 381.1 \ \text{nH/m}$
$C = 116.8 \ \text{pF/m}$
$Z_0 = 57.1 \ \Omega$
$n_{eff} = 2.00$