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Computational Methods in EE

Topic 1 – Errors in Computation

EE 4386/5301 Computational Methods in EE

Outline

Computational Methods in EE

- Errors in Computation
- Uncertainty Analysis

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Errors in Computation

Types of Errors

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Truncation Error – Arises when approximations are used instead of performing exact operations.

$$\frac{\partial f(x)}{\partial x} \approx \frac{f(x_b) - f(x_a)}{b-a} \qquad \sqrt{1+x} \approx 1 + \frac{x}{2}$$

Round-Off Error – Arises when limited significant figures are used to represent exact numbers

$$\pi, e, \sqrt{3}$$

```
>> 0.3/0.1 - 3

ans =
   -4.4409e-16
```

Human Error – Arises when people make mistakes.

$$1+1=3$$

Quantifying Error

Significant Figures

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Very often quantities are limited to some number of digits.

This can happen because a computer cannot store any more digits or because the measurement is not accurate out to that many digits.

Rules:
1. All non-zero digits are significant
2. Zeros that do nothing but place the decimal point are not significant.

1.234	4 significant digits
0.00545	3 significant digits
32100	3 or 5 significant digits (ambiguous)
0.0500	3 significant digits

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Quantifying Error

True Error

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True Error – Actual error in a measurement.

$$E_t = (\text{true error}) = (\text{true value}) - (\text{approximate value})$$

Example

The speed of light is exactly 299,792,458 m/s. An experiment was performed that measured the speed of light to be 299,792,445 m/s.

$$(\text{true value}) = 299,792,458 \text{ m/s}$$

$$(\text{approximate value}) = 299,792,445 \text{ m/s}$$

$$E_t = (\text{true error}) = (299,792,458 \text{ m/s}) - (299,792,445 \text{ m/s}) = 13 \text{ m/s}$$

Notes

It is very rare to know the true value or true error of a measurement.
True error does not convey the relative severity of the error.

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Quantifying Error

Relative Error

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Relative Error – Actual error in a measurement relative to the size of the measurement.

$$\varepsilon_t = (\text{relative error}) = \frac{(\text{true error})}{(\text{true value})}$$

Example

The speed of light is exactly 299,792,458 m/s. An experiment was performed that measured the speed of light to be 299,792,445 m/s.

$$(\text{true error}) = 13 \ \text{m/s}$$

$$(\text{true value}) = 299,792,458 \ \text{m/s}$$

$$(\text{relative error}) = \frac{13 \ \text{m/s}}{299,792,458 \ \text{m/s}} = 4.33 \times 10^{-8} \ \text{or} \ 0.00000433\%$$

Pretty good!

Notes

This is a much better metric to determining the severity of the error.

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Quantifying Error

Approximate Relative Error

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Approximate Relative Error – We rarely know the true value of something so we must calculate errors based on our approximation and approximate error.

$$\varepsilon_a = (\text{approximate relative error}) = \frac{(\text{approximate error})}{(\text{approximate value})}$$

Note

We can be sure that our number is correct to at least n significant digits if:

$$|\varepsilon_a| = \left(0.5 \times 10^{2-n}\right) \ \%$$

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Error Propagation

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In computation, we are forced to calculate things from quantities with some error. This results in some degree of error in our computations. error.

$$\tilde{x} = x + E_x$$

True error

True value

Our measured value with an error

How do we calculate the error E_f of $f(\tilde{x})$?

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Error Propagation

Taylor Series

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The Taylor series is very often used to determine error associated with truncation.

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

This is an infinite array. Suppose we retain only the first $n+1$ terms?

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + R_n$$

We have lumped all remaining terms into R_n.

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

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Error Propagation

Propagating Error

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First, we rewrite the Taylor series in terms of our measured value.

$$f(x) = f(\tilde{x}) + f'(\tilde{x})(x-\tilde{x}) + \frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2 + \cdots$$

Second, we bring to the left side of the equation.

$$\underbrace{f(x) - f(\tilde{x})}_{\text{This is the definition of error.}} = f'(\tilde{x})(x-\tilde{x}) + \frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2 + \cdots$$

Third, we ignore all higher order terms to get an expression for error.

$$f(x) - f(\tilde{x}) = f'(\tilde{x})(x-\tilde{x}) + \cancel{\frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2 + \cdots}$$

$$\boxed{\text{error} = \Delta f = f'(\tilde{x})\Delta x}$$

Extension to Two Variables

$$\Delta f = \Delta x\left|\frac{\partial f}{\partial x}\right| + \Delta y\left|\frac{\partial f}{\partial y}\right|$$

Error Propagation

Examples

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Example 1

Suppose we have $\tilde{x} = 2.0$ with an error of $E_x = 0.05$.

What is the error Δf if $f(\tilde{x}) = \tilde{x}^2$?

$$\Delta f(\tilde{x}) = f'(\tilde{x})\cdot\Delta\tilde{x} = 2\tilde{x}\cdot\Delta\tilde{x} = 2(2.0)\cdot 0.05 = 4\cdot 0.05 = \underline{0.2}$$

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Uncertainty Analysis

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What is Uncertainty?

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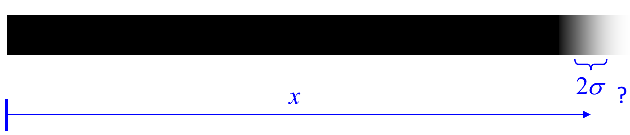
How long is this object?



We write our measurement as

$$x \pm \sigma$$

Since the measurement error is "blurred," we are not entirely certain about the error. The uncertainty σ is a statistically derived quantity.

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How to Interpret σ

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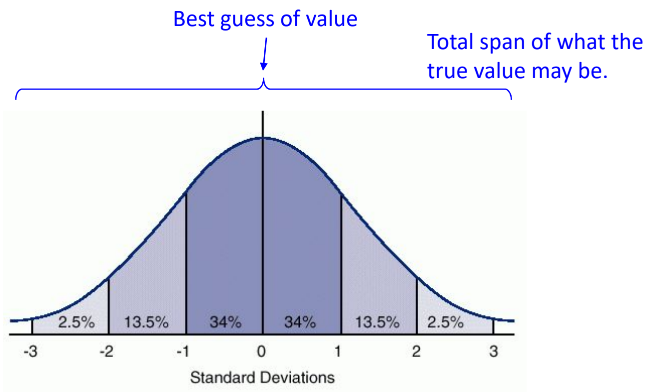

σ is one standard deviation.

There is a 68% chance that the error will be less than 1σ.
There is a 32% chance that the error will be greater than 1σ.

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Uncertainly Analysis

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Suppose we have some number of parameters, each with an associated uncertainty.

$$x \pm \sigma_x \qquad y \pm \sigma_y \qquad z \pm \sigma_z$$

Now suppose we calculate a new quantity from these.

$$f(x, y, z)$$

What is the uncertainty σ_f of $f(x,y,z)$?

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2$$

We call this propagating uncertainty through a calculation.

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Example 1

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What is the uncertainty of a decibel quantity?

$$10\log_{10}\left(x \pm \sigma_x\right)$$

<u>Solution</u>

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2$$

$$f(x) = 10\log_{10}(x)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[10\log_{10}(x)\right] = 10\left[\frac{\partial}{\partial x}\log_{10}(x)\right] = 10\frac{1}{x}\frac{1}{\ln 10} = \frac{10}{x\ln 10}$$

$$\sigma_f^2 = \left(\sigma_x \frac{10}{x\ln 10}\right)^2$$

$$\boxed{\sigma_f = 4.3429\frac{\sigma_x}{x}}$$

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Example 2

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What is the uncertainty of the sum of two quantities?

$$x + y$$

<u>Solution</u>

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2$$

$$f = x + y$$

$$\sigma_f^2 = \left[\sigma_x \frac{\partial}{\partial x}(x+y)\right]^2 + \left[\sigma_y \frac{\partial}{\partial y}(x+y)\right]^2 = \left[\sigma_x \cdot 1\right]^2 + \left[\sigma_y \cdot 1\right]^2$$

$$\boxed{\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}}$$

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Table of Uncertainty Calculations

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Function	Uncertainty
$f = ax$	$\sigma_f = a\sigma_x$
$f = ax \pm by$	$\sigma_f^2 = \left(a\sigma_x\right)^2 + \left(b\sigma_x\right)^2$
$f = xy$	$\left(\sigma_f/f\right)^2 = \left(\sigma_x/x\right)^2 + \left(\sigma_y/y\right)^2$
$f = x/y$	$\left(\sigma_f/f\right)^2 = \left(\sigma_x/x\right)^2 + \left(\sigma_y/y\right)^2$
$f = ax^{\pm b}$	$\sigma_f/f = b\sigma_x/x$
$f = a\ln\left(\pm bx\right)$	$\sigma_f = b\sigma_x/x$
$f = a\log x$	$\sigma_f = b\sigma_x/\left(x\ln 10\right)$
$f = ae^{\pm bx}$	$\sigma_f/f = b\sigma_x$
$f = a^{\pm bx}$	$\sigma_f/f = b\sigma_x \ln a$
$f = \sin x$	$\sigma_f = \sigma_x \cos x$
$f = \cos x$	$\sigma_f = \sigma_x \sin x$
$f = \tan x$	$\sigma_f = \sigma_x/\cos^2 x$
$f = \sin^{-1}\left(x\right)$	$\sigma_f^2 = \sigma_x^2/\left(1-x^2\right)$
$f = \cos^{-1}\left(x\right)$	$\sigma_f^2 = \sigma_x^2/\left(1-x^2\right)$
$f = \tan^{-1}\left(x\right)$	$\sigma_f = \sigma_x/\left(1+x^2\right)$

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