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Computational Methods in EE

# Topic 1 – Errors in Computation

*EE 4386/5301 Computational Methods in EE*

## Outline

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- Errors in Computation
- Uncertainty Analysis

Computational Methods in EE

# Errors in Computation



## Types of Errors

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**Truncation Error** – Arises when approximations are used instead of performing exact operations.

$$\frac{\partial f(x)}{\partial x} \approx \frac{f(x_b) - f(x_a)}{b - a} \qquad \sqrt{1+x} \approx 1 + \frac{x}{2}$$

**Round-Off Error** – Arises when limited significant figures are used to represent exact numbers

$$\pi, e, \sqrt{3}$$

```
>> 0.3/0.1 - 3

ans =
  -4.4409e-16
```

**Human Error** – Arises when people make mistakes.

$$1+1=3$$

# Quantifying Error
## *Significant Figures*

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Very often quantities are limited to some number of digits.

This can happen because a computer cannot store any more digits or because the measurement is not accurate out to that many digits.

Rules:
1. All non-zero digits are significant
2. Zeros that do nothing but place the decimal point are not significant.

How many significant digits?

| | |
|---|---|
| 1.234 | 4 significant digits |
| 0.00545 | 3 significant digits |
| 32100 | 3 or 5 significant digits (ambiguous) |
| 0.0500 | 3 significant digits |



# Quantifying Error
## *True Error*

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**True Error** – Actual error in a measurement.

$$\Delta = (\text{true error}) = (\text{true value}) - (\text{approximate value})$$

Example

The speed of light is exactly 299,792,458 m/s. An experiment was performed that measured the speed of light to be 299,792,445 m/s.

$$(\text{true value}) = 299,792,458 \text{ m/s}$$

$$(\text{approximate value}) = 299,792,445 \text{ m/s}$$

$$\Delta = (\text{true error}) = (299,792,458 \text{ m/s}) - (299,792,445 \text{ m/s}) = 13 \text{ m/s}$$

Notes
- It is very rare to know the true value or true error of a measurement.
- Unrealistic because the true value must be known, but good for analyzing how errors propagate through computations.
- True error does not convey the relative severity of the error.

# Quantifying Error

*Relative Error*

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**Relative Error** – Actual error in a measurement relative to the size of the measurement.

$$\varepsilon_{\mathrm{t}} = (\text{relative error}) = \frac{(\text{true error})}{(\text{true value})}$$

Example

The speed of light is exactly 299,792,458 m/s. An experiment was performed that measured the speed of light to be 299,792,445 m/s.

$$(\text{true error}) = 13 \ \text{m/s}$$

$$(\text{true value}) = 299,792,458 \ \text{m/s}$$

Pretty good!

$$(\text{relative error}) = \frac{13 \ \text{m/s}}{299,792,458 \ \text{m/s}} = 4.33\times10^{-8} = 0.0000000433 \xrightarrow{\times 100\%} 0.00000433\%$$

Notes

This is a much better metric to determining the severity of the error.



# Quantifying Error

*Approximate Relative Error*

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**Approximate Relative Error** – We rarely know the true value of something so we must calculate errors based on our approximation and approximate error.

$$\varepsilon_{\mathrm{a}} = (\text{approximate relative error}) = \frac{(\text{approximate error})}{(\text{approximate value})}$$

Note

We can be sure that our number is correct to at least $n$ significant digits if:

$$|\varepsilon_{\mathrm{a}}| = (0.5\times10^{2-n})\ \%$$

# Error Propagation

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In computation, we are forced to calculate things from quantities with some error. This results in some degree of error in our computations. error.

$$\tilde{x} = x + \Delta x$$

True error

True value

Our measured value with an error

How do we calculate the error $\Delta x$ of $f(\tilde{x})$?



# Error Propagation

*Taylor Series*

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The Taylor series is very often used to determine error associated with truncation.

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

This is an infinite array. Suppose we retain only the first $n+1$ terms?

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + R_n$$

We have lumped all remaining terms into $R_n$.

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

# Error Propagation

*Propagating Error*

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First, we rewrite the Taylor series in terms of our measured value.

$$f(x) = f(\tilde{x}) + f'(\tilde{x})(x-\tilde{x}) + \frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2 + \cdots$$

Second, we bring $f(\tilde{x})$ to the left side of the equation.

$$\underbrace{f(x) - f(\tilde{x})}_{\text{This is the definition of error } \Delta f \text{ of } f(\tilde{x}).} = f'(\tilde{x})(x-\tilde{x}) + \frac{f''(\tilde{x})}{2!}(x-\tilde{x})^2 + \cdots$$

These are $\Delta x$.

Third, we ignore all higher order terms to get an expression for error.

$$f(x) - f(\tilde{x}) = f'(\tilde{x})(\Delta x) + \cancel{\frac{f''(\tilde{x})}{2!}(\Delta x)^2} + \cdots$$

$$\boxed{\text{error} = \Delta f = f'(\tilde{x})\Delta x}$$

Generalization to Two Variables

$$\Delta f = \Delta x \left|\frac{\partial f}{\partial x}\right| + \Delta y \left|\frac{\partial f}{\partial y}\right|$$



# Error Propagation

*Examples*

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<u>Example 1</u>

Suppose we have $\tilde{x} = 2.0$ with an error of $\Delta x = 0.05$.

What is the error $\Delta f$ if $f(\tilde{x}) = \tilde{x}^2$?

$$\Delta f(\tilde{x}) = f'(\tilde{x}) \cdot \Delta x = 2\tilde{x} \cdot \Delta x = 2(2.0) \cdot 0.05 = 4 \cdot 0.05 = \underline{0.2}$$

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# Uncertainty Analysis



# What is Uncertainty?

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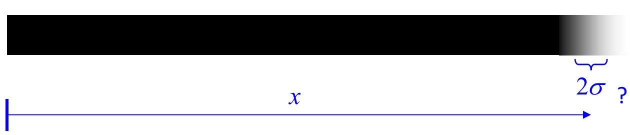
How long is this object?




We write our measurement as $x \pm \sigma$

Since the measurement error is "blurred," we are not entirely certain about the error. The uncertainty $\sigma$ is a statistically derived quantity.

Notes

- This is a more realistic way to treat error because we do not need to know the true value.
- This is used to predict errors in computations.

# How to Interpret $\sigma$

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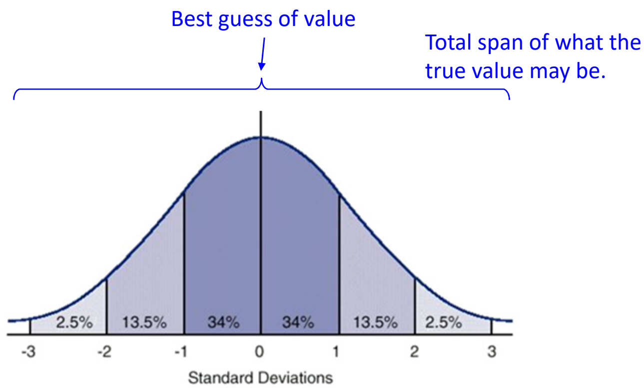


$\sigma$ is one standard deviation.

There is a 68% chance that the error will be less than $1\sigma$.
There is a 32% chance that the error will be greater than $1\sigma$.



# Uncertainly Analysis

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Suppose we have some number of parameters, each with an associated uncertainty.

$$x \pm \sigma_x \qquad y \pm \sigma_y \qquad z \pm \sigma_z$$

Now suppose we calculate a new quantity from these.

$$f(x, y, z)$$

What is the uncertainty $\sigma_f$ of $f(x,y,z)$?

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 + \left(\sigma_z \frac{\partial f}{\partial z}\right)^2$$

We call this propagating uncertainty through a calculation.

# Table of Uncertainty Calculations

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| Function | Uncertainty |
|---|---|
| $f = ax$ | $\sigma_f = a\sigma_x$ |
| $f = ax \pm by$ | $\sigma_f^2 = (a\sigma_x)^2 + (b\sigma_y)^2$ |
| $f = xy$ | $(\sigma_f/f)^2 = (\sigma_x/x)^2 + (\sigma_y/y)^2$ |
| $f = x/y$ | $(\sigma_f/f)^2 = (\sigma_x/x)^2 + (\sigma_y/y)^2$ |
| $f = x^{\pm b}$ | $\sigma_f/f = b\sigma_x/x$ |
| $f = \ln(\pm bx)$ | $\sigma_f = b\sigma_x/x$ |
| $f = \log x$ | $\sigma_f = b\sigma_x/(x \ln 10)$ |
| $f = e^{\pm bx}$ | $\sigma_f/f = b\sigma_x$ |
| $f = a^{\pm bx}$ | $\sigma_f/f = b\sigma_x \ln a$ |
| $f = \sin x$ | $\sigma_f = \sigma_x \cos x$ |
| $f = \cos x$ | $\sigma_f = \sigma_x \sin x$ |
| $f = \tan x$ | $\sigma_f = \sigma_x / \cos^2 x$ |
| $f = \sin^{-1}(x)$ | $\sigma_f^2 = \sigma_x^2/(1-x^2)$ |
| $f = \cos^{-1}(x)$ | $\sigma_f^2 = \sigma_x^2/(1-x^2)$ |
| $f = \tan^{-1}(x)$ | $\sigma_f = \sigma_x/(1+x^2)$ |



# Uncertainty Through Multiple Calculations

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Suppose we wish to find the uncertainty $\sigma_f$ when multiple calculations are involved.

$$f(a,b,c) = \ln(a+bc) \qquad \text{Given } \sigma_a,\ \sigma_b,\ \text{and } \sigma_c$$

| | Value | Uncertainty |
|---|---|---|
| 1 | $bc$ | $\left(\frac{\sigma_{bc}}{bc}\right)^2 = \left(\frac{\sigma_b}{b}\right)^2 + \left(\frac{\sigma_c}{c}\right)^2$ |
| 2 | $a+bc$ | $\sigma_{a+bc} = \sigma_a^2 + \sigma_{bc}^2$ |
| 3 | $\ln(a+bc)$ | $\sigma_{\ln(a+bc)} = \frac{\sigma_{a+bc}}{a+bc}$ |

# Example 1

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What is the uncertainty of a decibel quantity?

$$10\log_{10}\left(x \pm \sigma_x\right)$$

Solution

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2$$

$$f(x) = 10\log_{10}(x)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[10\log_{10}(x)\right] = 10\left[\frac{\partial}{\partial x}\log_{10}(x)\right] = 10\frac{1}{x}\frac{1}{\ln 10} = \frac{10}{x\ln 10}$$

$$\sigma_f^2 = \left(\sigma_x \frac{10}{x\ln 10}\right)^2$$

$$\boxed{\sigma_f = 4.3429\frac{\sigma_x}{x}}$$



# Example 2

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The signal-to-noise ratio (SNR) of a system is 50±1.2. What is the SNR in decibels along with the uncertainty?

Solution

The SNR in decibels is

$$10\log_{10}(50) = 16.99 \text{ dB}$$

The uncertainty is calculated using the equation derived in the previous example.

$$\sigma_f = 4.3429\frac{\sigma_x}{x} = 4.3429\frac{1.2}{50} = 0.1$$

The final answer is

$$\boxed{\text{SNR} = 16.99 \pm 0.1 \text{ dB}}$$

# Example 3

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What is the uncertainty of the sum of two quantities?

$$x + y$$

Solution

$$\sigma_f^2 = \left(\sigma_x \frac{\partial f}{\partial x}\right)^2 + \left(\sigma_y \frac{\partial f}{\partial y}\right)^2 \qquad f = x + y$$

$$\sigma_f^2 = \left[\sigma_x \frac{\partial}{\partial x}(x+y)\right]^2 + \left[\sigma_y \frac{\partial}{\partial y}(x+y)\right]^2 = \left[\sigma_x \cdot 1\right]^2 + \left[\sigma_y \cdot 1\right]^2$$

$$\boxed{\sigma_f = \sqrt{\sigma_x^2 + \sigma_y^2}}$$



# Example 4

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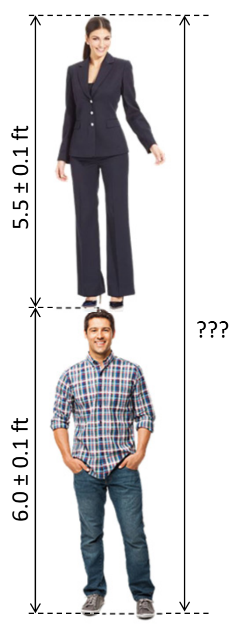
The height of person 1 was measured to be 6.0 ± 0.1 ft.

The height of person 2 was measured to be 5.5±0.1 ft.

If person 2 stands on the head of person 1, what is the total height and the uncertainty of the total height?




Solution

Total height

$$\begin{aligned} h &= h_1 + h_2 \\ &= (6.0\text{ ft}) + (5.5\text{ ft}) \\ &= 11.5\text{ ft} \end{aligned}$$

Uncertainty

$$\begin{aligned} \sigma_h &= \sqrt{\sigma_1^2 + \sigma_2^2} \\ &= \sqrt{(0.1\text{ ft})^2 + (0.1\text{ ft})^2} \\ &= 0.14\text{ ft} \end{aligned}$$

Final Answer

$$\boxed{h = 11.5 \pm 0.14\text{ ft}}$$