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Computational Methods in EE

Topic 6a – Numerical Integration

EE 4386/5301 Computational Methods in EE

Outline

Computational Methods in EE

- Introduction
- Discrete Integration
- Trapezoidal Integration
- Simpson's Integration
- Multiple Integrals
- Convergence

Computational Methods in EE

Introduction

Why Use Numerical Integration?

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How to we calculate the following integral?

$$\int_a^b e^{x^2}\,dx$$

No analytical solution exists to perform this integration by hand.

We are forced to calculate the integration by other means.

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Discrete Integration

Problem Setup

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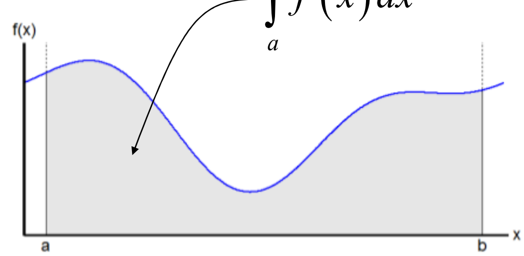
Suppose we have a function $f(x)$ and we want to integrate it from a to b.

$$\int_a^b f(x)\,dx$$



Solution (1 of 2)

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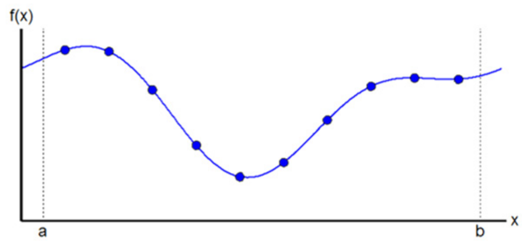
When we solve this problem on a computer, we will most likely only know the function value at discrete points.



Solution (2 of 2)

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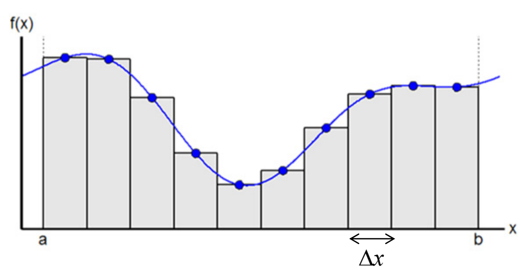
A simple approach to approximate this integral is to represent the area under this function as a series of rectangles.

Observe that the position of the points are at the center of the rectangles.



$$\int_a^b f(x)\,dx \approx \sum_{n=1}^{N} f(x_n)\Delta x = \frac{b-a}{N}\sum_{n=1}^{N} f(x_n) \qquad \Delta x = \frac{b-a}{N}$$

Error

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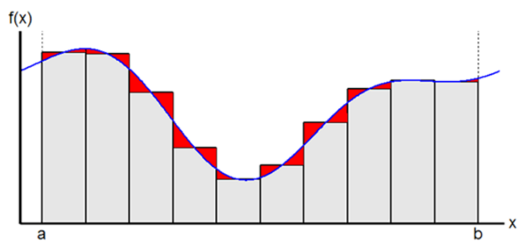
Approximating the integral this way produces some error.
Gaps between the true curve and the rectangles leads to error.



Reducing the Errors

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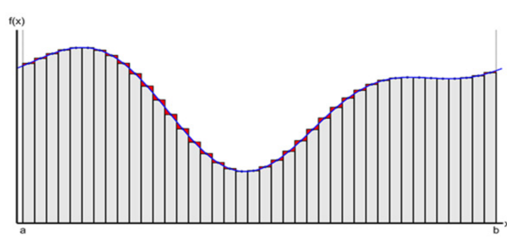
The only way to reduce error is to <u>use thinner rectangles</u>. However, this increases the number of computations that have to be performed which increases calculation time and could lead to larger round-off error.



Or, use a different numerical integration technique altogether!

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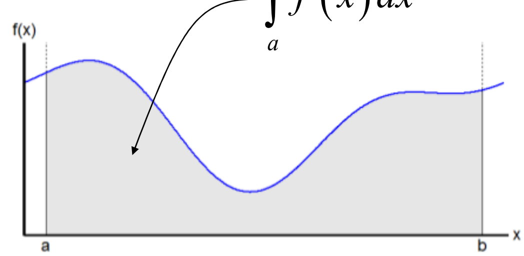
Trapezoidal Integration

Problem Setup

Computational Methods in EE

Suppose we have a function $f(x)$ and we want to integrate it from a to b.

$$\int_a^b f(x)\,dx$$



Solution (1 of 2)

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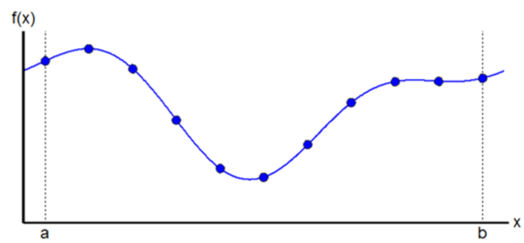
A more accurate technique for numerical integration uses the trapezoidal rule. For this, we place the points x_n differently.
Points are placed at the extreme ends and distributed evenly.



Solution (2 of 2)

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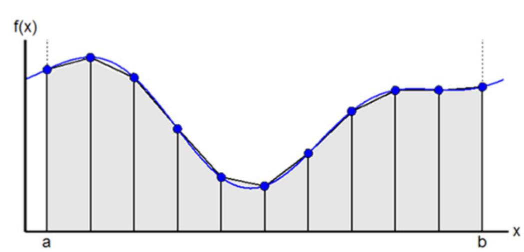
Instead of fitting rectangles under the curve, we use trapezoids. This conforms more closely to the curve.



Error

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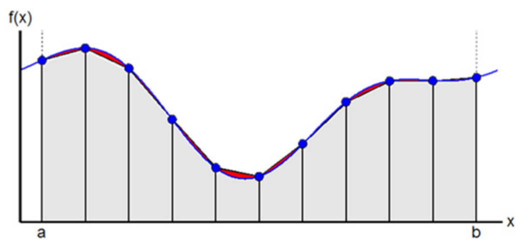
Approximating the integral this way still produces some error.
There is noticeably less error than with discrete integration.



The error for trapezoidal integration is $E_t(x) = -\frac{1}{12} f''(x)(b-a)^3$

Formulation (1 of 2)

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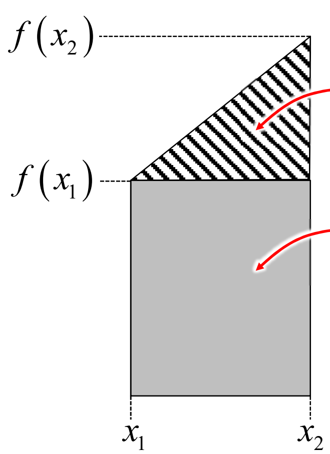
The total area of a trapezoid is



$$A_{tri} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(x_2 - x_1)(f_2 - f_1)$$

$$A_{rect} = (\text{width})(\text{height}) = (x_2 - x_1)(f_1)$$

<u>Total Area of Trapezoid</u>

$$\begin{aligned} A &= A_{rect} + A_{tri} \\ &= (x_2 - x_1) f_1 + \frac{1}{2}(x_2 - x_1)(f_2 - f_1) \\ &= (x_2 - x_1)\frac{f_1 + f_2}{2} \end{aligned}$$

Formulation (2 of 2)

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In trapezoidal integration, we add all of the areas of the trapezoids to approximate the integration.

$$\int_a^b f(x)\,dx \approx \begin{cases} \sum_{n=1}^{N} \left(x_{n+1} - x_n\right) \dfrac{f_n + f_{n+1}}{2} & \text{nonuniform spacing} \\ \dfrac{\Delta x}{2} \sum_{n=1}^{N} \left(f_n + f_{n+1}\right) & \text{uniform spacing} \end{cases}$$

Uniform Spacing

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When the spacing is uniform, trapezoidal integration reduces to

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2} \sum_{n=1}^{N} \left(f_n + f_{n+1}\right)$$

To understand this more deeply, we expand the summation over four trapezoids.

$$\sum_{n=1}^{N} \left(f_n + f_{n+1}\right) = \left(f_1 + f_2\right) + \left(f_2 + f_3\right) + \left(f_3 + f_4\right) + \left(f_4 + f_5\right)$$

We see that each point is included twice, except the two endpoints at $x = a$ and $x = b$.

$$\sum_{n=1}^{N} \left(f_n + f_{n+1}\right) = f_1 + 2f_2 + 2f_3 + 2f_4 + f_5$$

Discrete Vs. Trapezoidal Integration (1 of 2)

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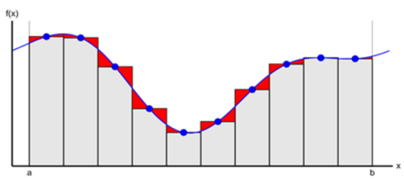
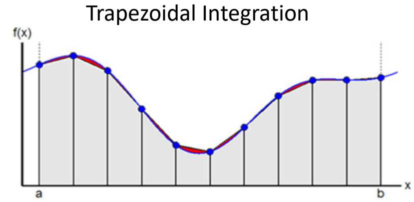
There are some key differences between discrete and trapezoidal integration:

- Points are distributed differently.
- Discrete integration is easier to implement.
- Trapezoidal integration has less error.
- Trapezoidal more elegantly handles nonuniform spacing.

Discrete Integration



Trapezoidal Integration



Discrete Vs. Trapezoidal Integration (2 of 2)

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Let's compare the equations for both discrete and trapezoidal integration. First, we can rearrange trapezoidal integration as follows:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\sum_{n=1}^{N}\left(f_n + f_{n+1}\right) = \Delta x\left(0.5 f_1 + f_2 + f_3 + f_4 + 0.5 f_5\right)$$

The equivalent equation for discrete integration is

$$\int_a^b f(x)\,dx \approx \Delta x \sum_{n=1}^{N} f_n = \Delta x\left(f_1 + f_2 + f_3 + f_4\right)$$

We see that trapezoidal integration reduces to discrete integration but with one extra rectangle added.

Interpreting Trapezoidal Integration as Discrete Integration

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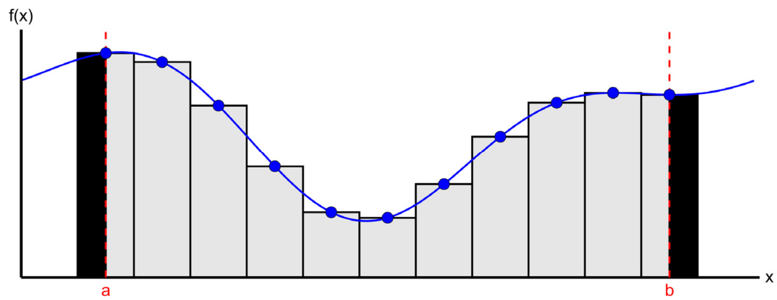
Trapezoidal integration can be written as

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\sum_{n=1}^{N}\left(f_n + f_{n+1}\right) = \Delta x\left(0.5f_1 + f_2 + f_3 + f_4 + 0.5f_5\right)$$

This can be interpreted as a modified discrete integration.



How Can Discrete & Trapezoidal Produce Roughly the Same Error?

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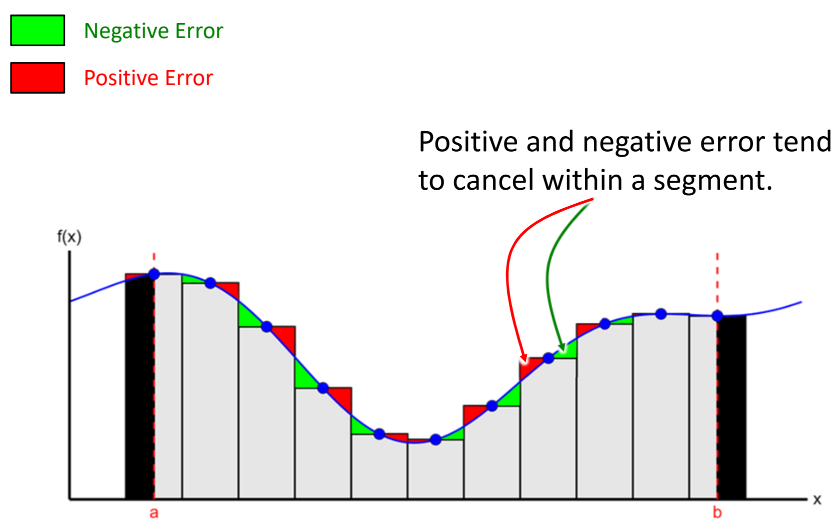

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Simpson's Integration

Simpson's 1/3 Rule

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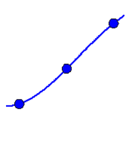
Suppose we have three adjacent points and we fit them to a second-order polynomial.



$$f(x) \cong a_0 + a_1 x + a_2 x^2$$

Now let's integrate the polynomial under the curve.

$$\int_{x_1}^{x_3} f(x)\,dx \approx \int_{x_1}^{x_3} \left(a_0 + a_1 x + a_2 x^2\right) dx$$

$$\approx \frac{1}{3}\Delta x\left(f_1 + 4f_2 + f_3\right)$$

To implement Simpson's 1/3 rule, we simply apply this to $f(x)$ in groups of 3 points.

Derivation of Simpson's 1/3 Rule

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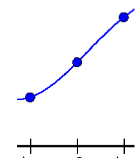
First, fit the three points to a polynomial.

$$f(x) \cong a_0 + a_1 x + a_2 x^2$$

$$a_0 = f_2 \qquad a_1 = \frac{f_3 - f_1}{2\Delta x} \qquad a_2 = \frac{f_3 - 2f_2 + f_1}{2(\Delta x)^2}$$



Second, integrate the polynomial from $-\Delta x$ to Δx.

$$\int_{-\Delta x}^{\Delta x} \left(a_0 + a_1 x + a_2 x^2\right) dx = \left(a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3\right)\Bigg|_{-\Delta x}^{\Delta x} = 2a_0 \Delta x + \frac{2}{3} a_2 (\Delta x)^3$$

Substitute in the expressions for a_0, a_1, and a_2.

$$2a_0 \Delta x + \frac{2}{3} a_2 (\Delta x)^3 = 2 f_2 \Delta x + \frac{2}{3} \frac{f_3 - 2f_2 + f_1}{2(\Delta x)^2} (\Delta x)^3 = \frac{1}{3} \Delta x \left(f_1 + 4f_2 + f_3\right)$$

Implementation of Simpson's 1/3 Rule

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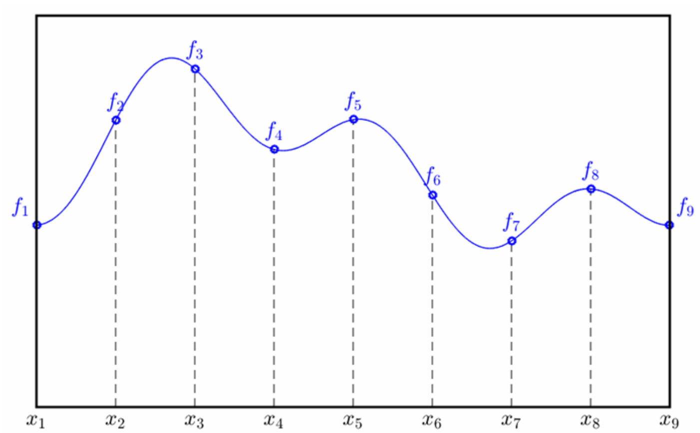
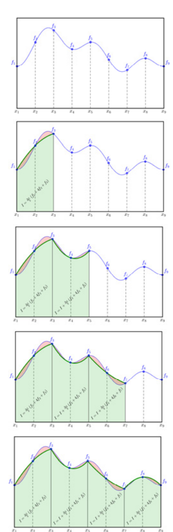
Animation of Numerical Integration Using Simpson's 1/3 Rule





Simpson's 3/8 Rule

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This is similar to Simpson's 1/3 rule, except we apply it to $f(x)$ in groups of 4 points.

$$\int_{x_1}^{x_4} f(x)\,dx \approx \frac{3}{8}\Delta x\left(f_1 + 3f_2 + 3f_3 + f_4\right)$$

Computational Methods in EE

Multiple Integrals

Problem Setup

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Suppose we have a function $f(x,y)$ with two independent variables.

How do we evaluate a double integral?

$$\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dxdy = ?$$

We will view this as an "integral of integrals."

$$\int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)dx = ?$$

We evaluate the inside integral for each step of in the integration of the outside integral.

Illustration of Numerical Double Integration

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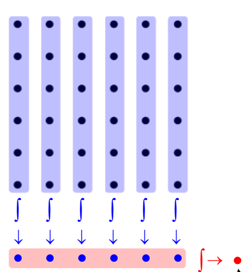
We start with a 2D array.

We numerically integrate each of the columns to get a 1D array.

We numerically integrate the 1D array.



Our final answer is the numerical double integral of the original 2D array.

Via Discrete Integration

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This is very easy using discrete integration. Our discrete equation is

$$\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dxdy \approx \sum_{m=0}^{M}\sum_{n=0}^{N} f(a+m\Delta x, c+n\Delta y)\Delta x\Delta y$$

$$\Delta x = \frac{b-a}{M}$$

$$\Delta y = \frac{d-c}{N}$$

The MATLAB code to do this is simply

```
dx = (b - a)/M;
dy = (d - c)/N;
I  = sum(f(:))*dx*dy;
```


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Convergence

What is Convergence?

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Convergence is the tendency of a numerical algorithm to approach a specific value as the resolution of the algorithm is increased.

This does NOT imply the answer gets more correct.

There may still be something wrong with your calculation!

Demonstration of Convergence

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Suppose we wish to evaluate the following integral:

$$\int_0^{\pi} \sin x dx$$

How many segments are necessary?

There is no way to tell. We must perform a convergence study!

Analytical Answer

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To check ourselves, we can solve the integral analytically…

$$\int_0^{\pi} \sin x dx = -\cos x\Big|_0^{\pi}$$

$$= (-\cos\pi) - (-\cos 0)$$

$$= 2$$

Convergence Study

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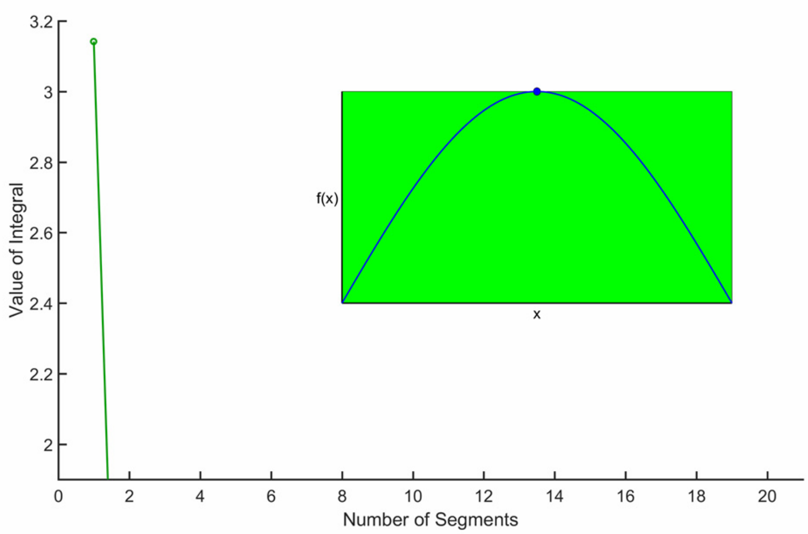

Convergence Study

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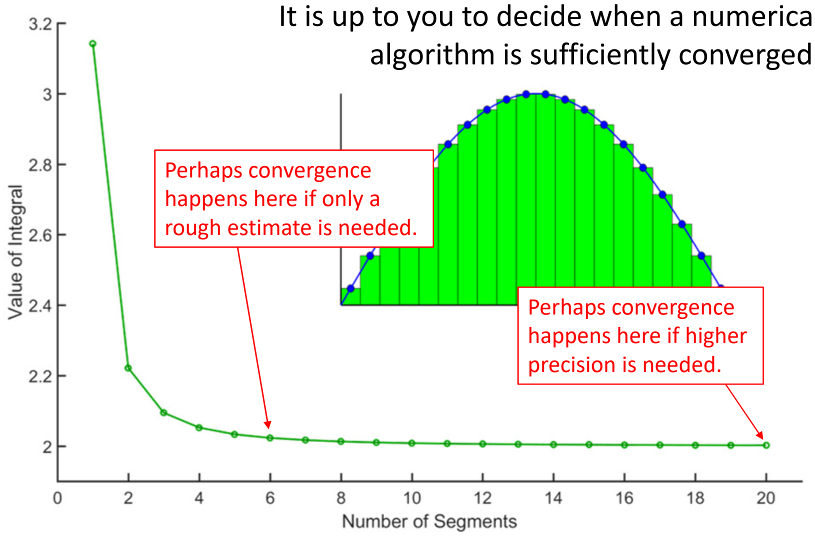
It is up to you to decide when a numerical algorithm is sufficiently converged.



Convergence Does NOT Imply Correctness

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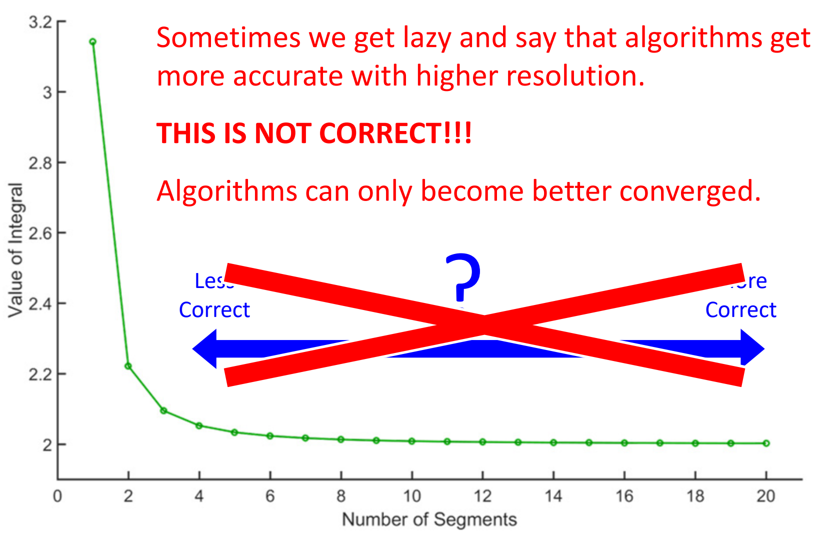
Sometimes we get lazy and say that algorithms get more accurate with higher resolution.

THIS IS NOT CORRECT!!!

Algorithms can only become better converged.



Rule-of-Thumb for Resolution

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For calculations involving waves, the resolution begins to converge when you resolve one wave cycle with about 10 divisions.

$$\Delta \approx \frac{\lambda}{10} \qquad \lambda \equiv \text{wavelength} \qquad \rightarrow$$