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Computational Methods in EE

Topic 7b – 1D Finite-Difference Method

EE 4386/5301 Computational Methods in EE

Outline

Computational Methods in EE

- 1D Derivative Matrices
- Incorporating Boundary Conditions
- Solving ODE's

Computational Methods in EE

1D Derivative Matrices

Placing Diagonals in Sparse Matrices in MATLAB

Computational Methods in EE

```
M = 6;
Z = sparse(M,M);
d = ones(M,1);
A = spdiags(d,0,Z);
```

⇨

```
A =
[ 1  0  0  0  0  0  ]
[ 0  1  0  0  0  0  ]
[ 0  0  1  0  0  0  ]
[ 0  0  0  1  0  0  ]
[ 0  0  0  0  1  0  ]
[ 0  0  0  0  0  1  ]
```

```
M = 6;
Z = sparse(M,M);
d = ones(M,1);
A = spdiags(-d,-1,Z);
A = spdiags(+d,+1,A);
```

⇨

```
A =
[  0   1   0   0   0   0  ]
[ -1   0   1   0   0   0  ]
[  0  -1   0   1   0   0  ]
[  0   0  -1   0   1   0  ]
[  0   0   0  -1   0   1  ]
[  0   0   0   0  -1   0  ]
```

Computational Methods in EE

Incorporating Boundary Conditions

Dirichlet Boundary Conditions

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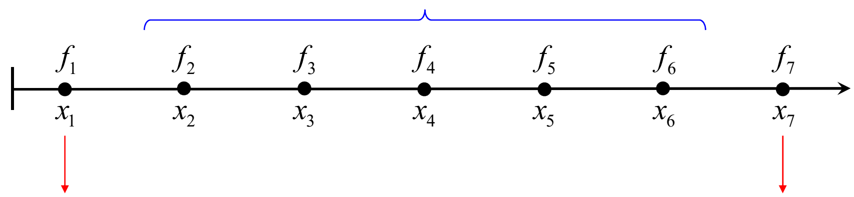
The simplest boundary condition is to assume all function values outside of the grid are zero.

$$\frac{d^2 f_i}{dx^2} \cong \frac{f_{i-1} - 2f_i + f_{i+1}}{(\Delta x)^2}$$

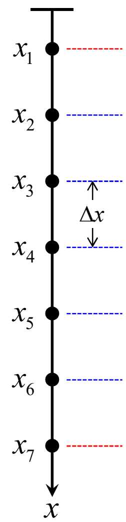


$$\frac{d^2 f_1}{dx^2} \cong \frac{0 - 2f_1 + f_2}{(\Delta x)^2}$$

$$\frac{d^2 f_7}{dx^2} \cong \frac{f_6 - 2f_7 + 0}{(\Delta x)^2}$$

Dirichlet Boundary Conditions (2 of 2)

Computational Methods in EE



$$\frac{d^2 f_1}{dx^2} \cong \frac{0 - 2f_1 + f_2}{(\Delta x)^2}$$

$$\frac{d^2 f_2}{dx^2} \cong \frac{f_1 - 2f_2 + f_3}{(\Delta x)^2}$$

$$\frac{d^2 f_3}{dx^2} \cong \frac{f_2 - 2f_3 + f_4}{(\Delta x)^2}$$

$$\frac{d^2 f_4}{dx^2} \cong \frac{f_3 - 2f_4 + f_5}{(\Delta x)^2}$$

$$\frac{d^2 f_5}{dx^2} \cong \frac{f_4 - 2f_5 + f_6}{(\Delta x)^2}$$

$$\frac{d^2 f_6}{dx^2} \cong \frac{f_5 - 2f_6 + f_7}{(\Delta x)^2}$$

$$\frac{d^2 f_7}{dx^2} \cong \frac{f_6 - 2f_7 + 0}{(\Delta x)^2}$$

$$\Rightarrow \underbrace{\frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}}_{\left[D_x^2\right]} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Periodic Boundary Conditions (1 of 2)

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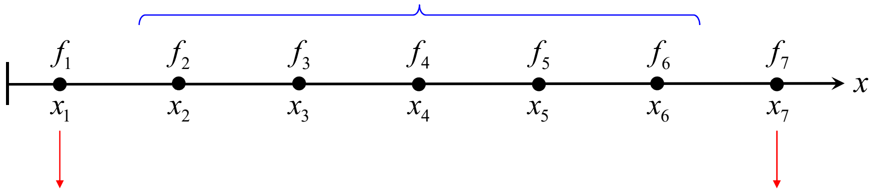
If the problem is periodic (i.e. keeps repeating), then the value outside of the grid is the same as the value at the opposite side of the grid.

$$\frac{d^2 f_i}{dx^2} \cong \frac{f_{i-1} - 2f_i + f_{i+1}}{(\Delta x)^2}$$

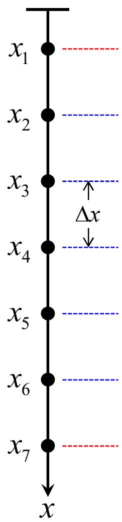


$$\frac{d^2 f_1}{dx^2} \cong \frac{f_7 - 2f_1 + f_2}{(\Delta x)^2}$$

$$\frac{d^2 f_7}{dx^2} \cong \frac{f_6 - 2f_7 + f_1}{(\Delta x)^2}$$

Periodic Boundary Conditions (2 of 2)

Computational Methods in EE



$$\frac{d^2 f_1}{dx^2} \cong \frac{f_7 - 2f_1 + f_2}{(\Delta x)^2}$$
$$\frac{d^2 f_2}{dx^2} \cong \frac{f_1 - 2f_2 + f_3}{(\Delta x)^2}$$
$$\frac{d^2 f_3}{dx^2} \cong \frac{f_2 - 2f_3 + f_4}{(\Delta x)^2}$$
$$\frac{d^2 f_4}{dx^2} \cong \frac{f_3 - 2f_4 + f_5}{(\Delta x)^2}$$
$$\frac{d^2 f_5}{dx^2} \cong \frac{f_4 - 2f_5 + f_6}{(\Delta x)^2}$$
$$\frac{d^2 f_6}{dx^2} \cong \frac{f_5 - 2f_6 + f_7}{(\Delta x)^2}$$
$$\frac{d^2 f_7}{dx^2} \cong \frac{f_6 - 2f_7 + f_1}{(\Delta x)^2}$$

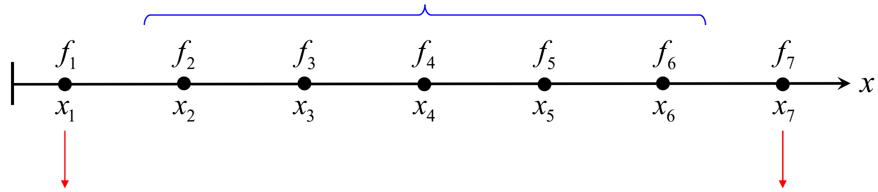
$$\Rightarrow \underbrace{\frac{1}{(\Delta x)^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}}_{[D_x^2]} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Neuman Boundary Conditions (1 of 3)

Computational Methods in EE

The Neuman boundary condition allows functions to continue linearly off of the grid as if to infinity.

$$\frac{df_i}{dx} \cong \frac{f_{i+1} - f_{i-1}}{2\Delta x} \qquad \frac{d^2 f_i}{dx^2} \cong \frac{f_{i-1} - 2f_i + f_{i+1}}{(\Delta x)^2}$$



$$\frac{df_1}{dx} \cong \frac{f_2 - f_1}{\Delta x} \qquad \frac{d^2 f_1}{dx^2} \cong 0$$

$$\frac{df_7}{dx} \cong \frac{f_7 - f_6}{\Delta x} \qquad \frac{d^2 f_7}{dx^2} \cong 0$$

Neuman Boundary Conditions (2 of 3)

Computational Methods in EE

x_1: $\dfrac{df_1}{dx} \cong \dfrac{2f_2 - 2f_1}{2\Delta x}$

x_2: $\dfrac{df_2}{dx} \cong \dfrac{f_3 - f_1}{2\Delta x}$

x_3: $\dfrac{df_3}{dx} \cong \dfrac{f_4 - f_2}{2\Delta x}$

Δx

x_4: $\dfrac{df_4}{dx} \cong \dfrac{f_5 - f_3}{2\Delta x}$

x_5: $\dfrac{df_5}{dx} \cong \dfrac{f_6 - f_4}{2\Delta x}$

x_6: $\dfrac{df_6}{dx} \cong \dfrac{f_7 - f_5}{2\Delta x}$

x_7: $\dfrac{df_7}{dx} \cong \dfrac{2f_7 - 2f_6}{2\Delta x}$

x

$$\Rightarrow \underbrace{\frac{1}{2\Delta x}\begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}}_{[D_x]} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Neuman Boundary Conditions (3 of 3)

Computational Methods in EE

x_1: $\dfrac{d^2 f_1}{dx^2} \cong 0$

x_2: $\dfrac{d^2 f_2}{dx^2} \cong \dfrac{f_1 - 2f_2 + f_3}{(\Delta x)^2}$

x_3: $\dfrac{d^2 f_3}{dx^2} \cong \dfrac{f_2 - 2f_3 + f_4}{(\Delta x)^2}$

Δx

x_4: $\dfrac{d^2 f_4}{dx^2} \cong \dfrac{f_3 - 2f_4 + f_5}{(\Delta x)^2}$

x_5: $\dfrac{d^2 f_5}{dx^2} \cong \dfrac{f_4 - 2f_5 + f_6}{(\Delta x)^2}$

x_6: $\dfrac{d^2 f_6}{dx^2} \cong \dfrac{f_5 - 2f_6 + f_7}{(\Delta x)^2}$

x_7: $\dfrac{d^2 f_7}{dx^2} \cong 0$

x

$$\Rightarrow \underbrace{\frac{1}{(\Delta x)^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{[D_x^2]} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

High-Order Boundary Conditions (1 of 2)

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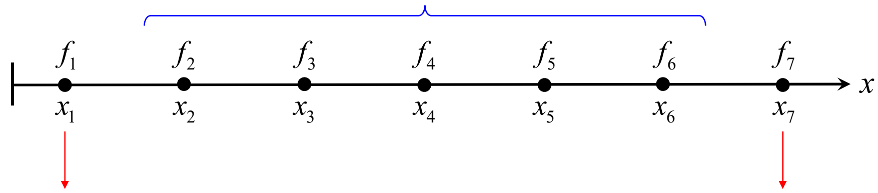
Here we estimate the derivative at the boundaries using special finite-difference equations derives specifically for these points.

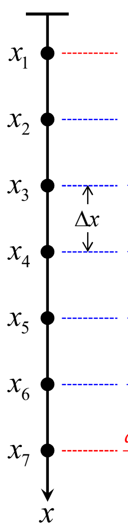
$$\frac{d^2 f_i}{dx^2} \cong \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2}$$



$$\frac{d^2 f_1}{dx^2} \cong \frac{2f_1 - 5f_2 + 4f_3 - f_4}{h^2} \qquad \frac{d^2 f_7}{dx^2} \cong \frac{-f_4 + 4f_5 - 5f_6 + 2f_7}{h^2}$$

High-Order Boundary Conditions (2 of 2)

Computational Methods in EE



$$\frac{d^2 f_1}{dx^2} \cong \frac{2f_1 - 5f_2 + 4f_3 - f_4}{(\Delta x)^2}$$

$$\frac{d^2 f_2}{dx^2} \cong \frac{f_1 - 2f_2 + f_3}{(\Delta x)^2}$$

$$\frac{d^2 f_3}{dx^2} \cong \frac{f_2 - 2f_3 + f_4}{(\Delta x)^2}$$

$$\frac{d^2 f_4}{dx^2} \cong \frac{f_3 - 2f_4 + f_5}{(\Delta x)^2}$$

$$\frac{d^2 f_5}{dx^2} \cong \frac{f_4 - 2f_5 + f_6}{(\Delta x)^2}$$

$$\frac{d^2 f_6}{dx^2} \cong \frac{f_5 - 2f_6 + f_7}{(\Delta x)^2}$$

$$\frac{d^2 f_7}{dx^2} \cong \frac{-f_4 + 4f_5 - 5f_6 + 2f_7}{(\Delta x)^2}$$

$$\Rightarrow \underbrace{\frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -5 & 4 & -1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -1 & 4 & -5 & 2 \end{bmatrix}}_{\left[D_x^2\right]} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

Solving Ordinary Differential Equations

Boundary Values

Many differential equations give boundary values or initial conditions that must be incorporated into the matrix equation in order to obtain a numerical solution.

For example, a problem may be stated as…

$$\frac{d^2 f(x)}{dx^2} + 5\frac{df(x)}{dx} + 6f(x) = 0 \qquad 0 \le x \le 2$$

$$f(0) = 2 \qquad f(2) = 0.2$$

Boundary values

Why are Boundary Values Needed?

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To solve our differential equation, we first convert it to matrix form.

$$\frac{d^2}{dx^2} f(x) + 5\frac{d}{dx} f(x) + 6 f(x) = 0$$

$$\left[D_x^2\right]\left[f\right] + 5\left[D_x\right]\left[f\right] + 6\left[f\right] = \left[0\right]$$

Next, this is rearranged into standard form.

$$\left[A\right]\left[f\right] = \left[0\right] \qquad \left[A\right] = \left[D_x^2\right] + 5\left[D_x\right] + 6\left[I\right]$$

This is not solvable.

$$\left[f\right] = \left[A\right]^{-1}\left[0\right] = \left[0\right]$$

We need to incorporate the boundary values in order to obtain a nontrivial solution. When we do this, our matrix equation becomes

$$\left[B\right]\left[f\right] = \left[b\right]$$

Incorporating Boundary Values

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It is possible to incorporate the boundary values into the individual matrices of the final matrix equation. However, it is usually easier to incorporate these into the final matrix equation.