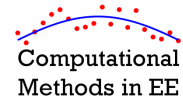




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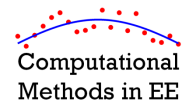


Topic 7c – Numerical Analysis of Slab Waveguides

EE 4386/5301 Computational Methods in EE

Outline

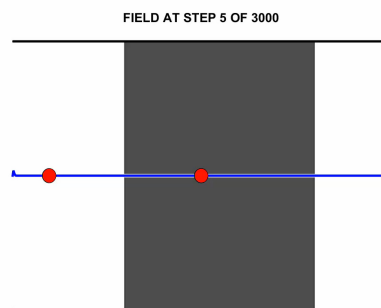
- Slab Waveguides
- Formulation
- Implementation



Slab Waveguides

Refractive Index n

Light travels at different speeds when it is inside different materials.

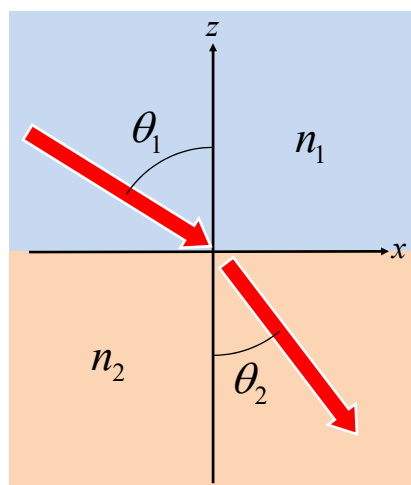
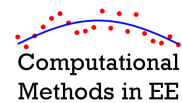


Frequency is constant.
Speed changes.
Wavelength changes.

The factor by which light slows down is called the *refractive index*.

$$n = \frac{c}{v}$$

Snell's Law



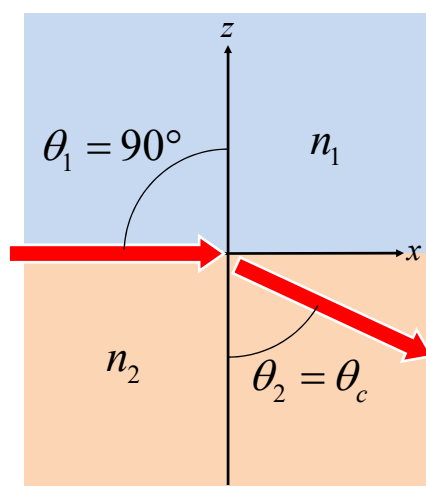
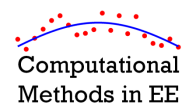
Snell's law quantifies the angles of light rays at an interface.

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Slab Waveguide Analysis

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Critical Angle



There exists a special angle, the *critical angle*, where the ray in the low-index medium is at 90° .

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

$$n_1 \sin 90^\circ = n_2 \sin \theta_c$$

$$n_1 = n_2 \sin \theta_c$$

$$\sin \theta_c = n_1/n_2$$

$$\theta_c = \sin^{-1}(n_1/n_2)$$

$$\theta_c = \sin^{-1}(n_1/n_2)$$

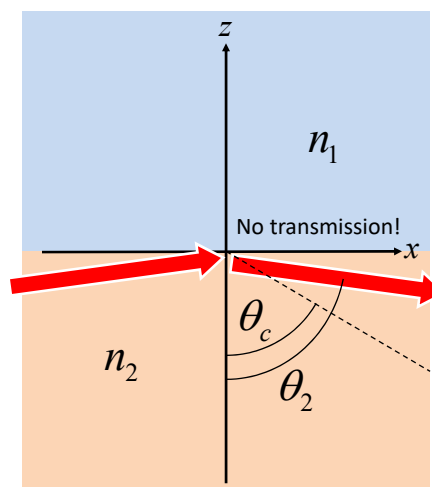
where $n_2 > n_1$

Slab Waveguide Analysis

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Total Internal Reflection (TIR)

Computational
Methods in EE



When a light ray is incident onto an interface at an angle greater than the critical angle, the light completely reflects and no light is transmitted.

This is called *total internal reflection* (TIR).

$$\theta_2 > \theta_c$$

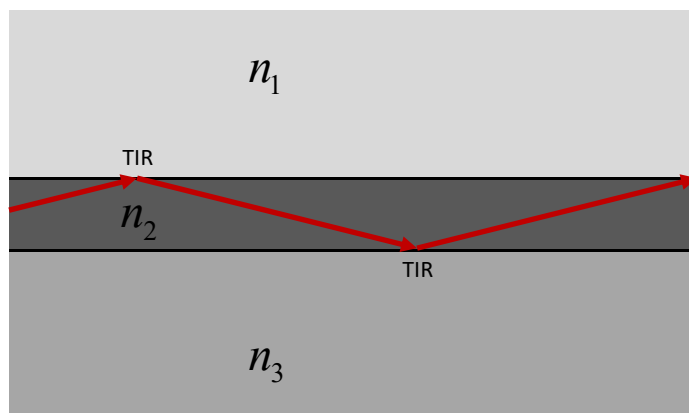
Slab Waveguide Analysis

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The Slab Waveguide

Computational
Methods in EE

If we “sandwich” a slab of high-index material between two materials with lower refractive index, we form a slab waveguide.

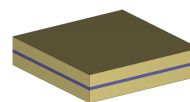


Conditions

$$n_2 > n_1$$

and

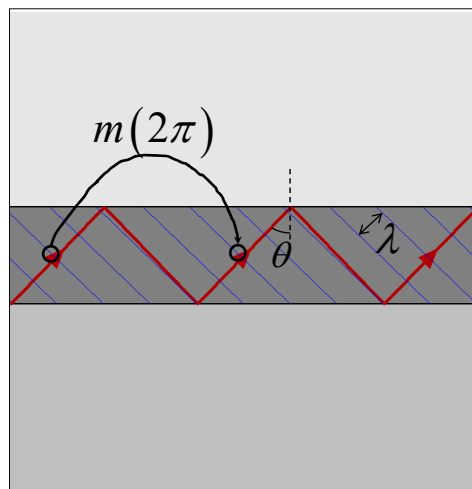
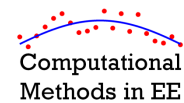
$$n_2 > n_3$$



Slab Waveguide Analysis

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Ray Tracing Picture



The round trip phase of a ray must be an integer multiple of 2π . Otherwise the wave will interfere with itself and escape from the slab.

Because of this, only certain angles are allowed to propagate in the waveguide.

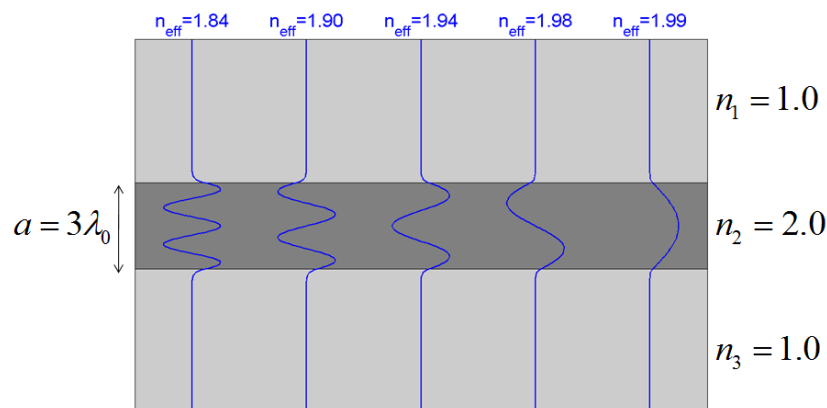
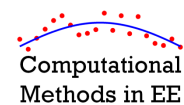
This is the origin of discrete modes in a waveguide.

$$\beta = k_0 n_{\text{eff}} = k_0 n \sin \theta$$

Slab Waveguide Analysis

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Rigorous Analysis



$$\beta = k_0 n_{\text{eff}} = k_0 n \sin \theta$$

Slab Waveguide Analysis

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Mathematical Form of Solution

Computational Methods in EE

$$\vec{E}(x, y, z) = \vec{A}(x) e^{-j\beta z}$$

Amplitude Profile

Wave oscillations

$\beta \equiv$ propagation constant

Slab Waveguide Analysis

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Slab Vs. Channel Waveguides

Computational Methods in EE

Slab waveguides confine energy in only one transverse direction.

Channel waveguides confine energy in both transverse directions.

Slab Waveguide Analysis

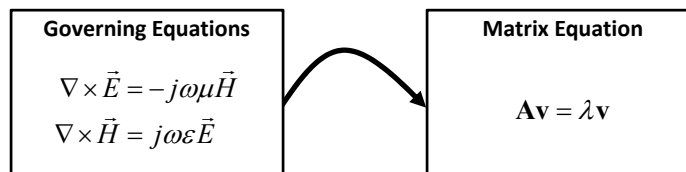
Slide 12

Formulation

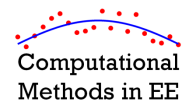
What is Formulation?

Formulation is the initial analytical work we do before implementing a computer code.

Usually we start with the governing equation(s) and end with the matrix equation to be solved.



Governing Equations



Since this is an electromagnetic problem, we start with Maxwell's curl equations.

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

Vector Curl

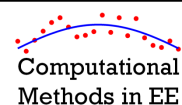
The curl of a vector is a measure of the vector field's tendency to circulate about an axis. The curl quantity is directly along this axis and the magnitude measures the strength of the circulation.

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{a}_z$$

Slab Waveguide Analysis

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Expand Governing Equations (1 of 2)



If we expand the first equation into its vector components, we get

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{a}_z = -j\omega\mu(H_x \hat{a}_x + H_y \hat{a}_y + H_z \hat{a}_z)$$

The vector components on each side must be equal.

$$x\text{-component: } \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$y\text{-component: } \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$z\text{-component: } \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

Slab Waveguide Analysis

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Expand Governing Equations (2 of 2)

Computational
Methods in EE

We now have

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z$$

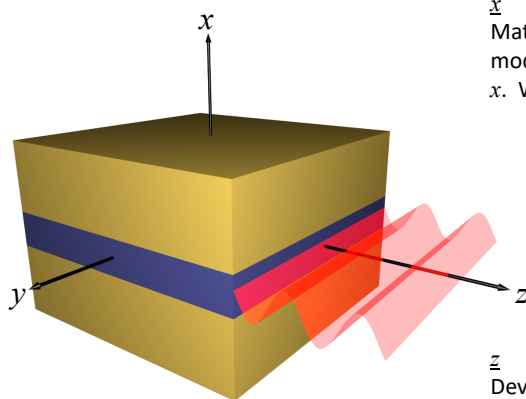
Slab Waveguide Analysis

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How to Reduce Dimensions

Computational
Methods in EE

It is always good practice to minimize the number of dimensions utilized in a numerical analysis.



\underline{x}
Material changes as a function of x . The mode profile will change as a function of x . We must retain this dimension.

\underline{y}
Device is uniform. Wave does not propagate in this direction. Mode profile is uniform.

$$\frac{\partial}{\partial y} = 0$$

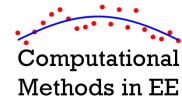
\underline{z}
Device is uniform. Wave propagates in this direction so wave phase is increasing.

$$\frac{\partial}{\partial z} = -j\beta$$

Slab Waveguide Analysis

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Apply $\partial/\partial y = 0$



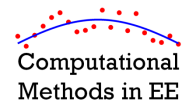
Since nothing is changing in the y direction, any derivative with respect to y must be zero.

$$\begin{array}{ccc}
 \cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x & & -\frac{\partial E_y}{\partial z} = -j\omega\mu H_x \\
 \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y & \longrightarrow & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \\
 \frac{\partial E_y}{\partial x} - \cancel{\frac{\partial E_x}{\partial y}} = -j\omega\mu H_z & & \frac{\partial E_y}{\partial x} = -j\omega\mu H_z \\
 \\
 \cancel{\frac{\partial H_z}{\partial y}} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x & & -\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \\
 \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y & \longrightarrow & \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \\
 \frac{\partial H_y}{\partial x} - \cancel{\frac{\partial H_x}{\partial y}} = j\omega\varepsilon E_z & & \frac{\partial H_y}{\partial x} = j\omega\varepsilon E_z
 \end{array}$$

Slab Waveguide Analysis

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Two Distinct Mode Types



Our revised governing equations have separated into two distinct mode types.

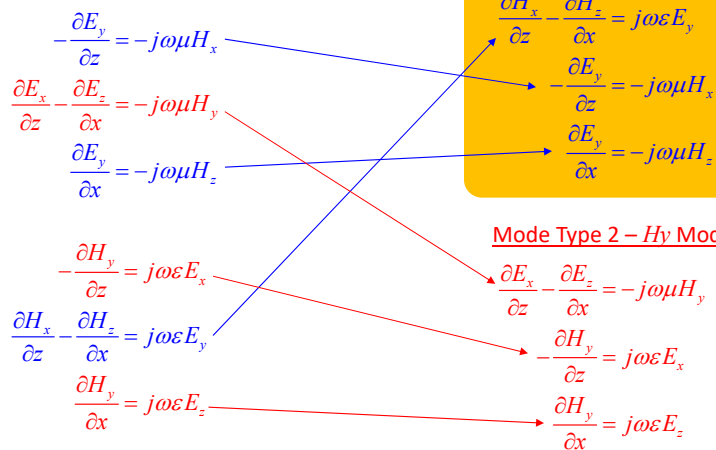
We will analyze the E_y mode

Mode Type 1 - E_y Mode

$$\begin{array}{l}
 \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \\
 -\frac{\partial E_y}{\partial z} = -j\omega\mu H_x \\
 \frac{\partial E_y}{\partial x} = -j\omega\mu H_z
 \end{array}$$

Mode Type 2 - H_y Mode

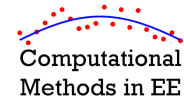
$$\begin{array}{l}
 \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \\
 -\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \\
 \frac{\partial H_y}{\partial x} = j\omega\varepsilon E_z
 \end{array}$$



Slab Waveguide Analysis

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What About $\partial/\partial z$?



Our waveguide mode has the following mathematical form

$$\vec{E}(x, y, z) = \vec{A}(x) e^{-j\beta z}$$

Let's calculate the partial derivative with respect to z and see what happens.

$$\begin{aligned} \frac{\partial}{\partial z} \vec{E}(x, y, z) &= \frac{\partial}{\partial z} [\vec{A}(x) e^{-j\beta z}] = \vec{A}(x) \frac{\partial}{\partial z} e^{-j\beta z} + e^{-j\beta z} \frac{\partial}{\partial z} \vec{A}(x) \\ &= -j\beta \underbrace{\vec{A}(x) e^{-j\beta z}}_{\vec{E}(x, y, z)} = -j\beta \vec{E}(x, y, z) \end{aligned}$$

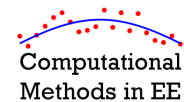
We conclude that

$$\frac{\partial}{\partial z} = -j\beta$$

Slab Waveguide Analysis

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1D Governing Equations



The equations for the E_y mode were

$$\begin{aligned} \frac{\partial}{\partial z} H_x - \frac{\partial H_z}{\partial x} &= j\omega\epsilon E_y \\ -\frac{\partial}{\partial z} E_y &= -j\omega\mu H_x \\ \frac{\partial E_y}{\partial x} &= -j\omega\mu H_z \end{aligned}$$

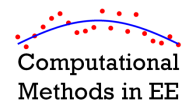
We simply replace $\partial/\partial z$ with $-j\beta$.

$$\begin{aligned} -j\beta H_x - \frac{\partial H_z}{\partial x} &= j\omega\epsilon E_y \\ j\beta E_y &= -j\omega\mu H_x \\ \frac{\partial E_y}{\partial x} &= -j\omega\mu H_z \end{aligned}$$

Slab Waveguide Analysis

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Normalize



Before converting our equations to matrix form, we should normalize the spatial coordinate x to put it in terms of wavelength in some manner.

$$\tilde{x} = \frac{x}{\lambda_0}$$

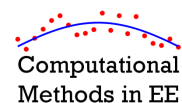
It will be mathematically convenient to normalize by multiplying x by the free space wave number k_0 .

$$\tilde{x} = k_0 x \quad k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega\sqrt{\mu\epsilon}}{n}$$

Slab Waveguide Analysis

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Normalized Equations



Applying our normalization, our equations become

$$\begin{aligned} -jn_{\text{eff}} H_x - \frac{\partial H_z}{\partial \tilde{x}} &= j\sqrt{\frac{\epsilon_0}{\mu_0}} \epsilon_r E_y \\ jn_{\text{eff}} E_y &= -j\sqrt{\frac{\mu_0}{\epsilon_0}} \mu_r H_x \\ \frac{\partial E_y}{\partial \tilde{x}} &= -j\sqrt{\frac{\mu_0}{\epsilon_0}} \mu_r H_z \end{aligned}$$

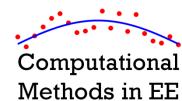
Last, at optical frequencies, the permeability is negligible so $\mu_r = 1$.

$$\begin{aligned} -jn_{\text{eff}} H_x - \frac{\partial H_z}{\partial \tilde{x}} &= j\sqrt{\frac{\epsilon_0}{\mu_0}} \epsilon_r E_y \\ jn_{\text{eff}} E_y &= -j\sqrt{\frac{\mu_0}{\epsilon_0}} H_x \\ \frac{\partial E_y}{\partial \tilde{x}} &= -j\sqrt{\frac{\mu_0}{\epsilon_0}} H_z \end{aligned}$$

Slab Waveguide Analysis

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Final Governing Equation



We solve the last two equations for H_x and H_z .

$$jn_{\text{eff}}E_y = -j\sqrt{\frac{\mu_0}{\epsilon_0}}H_x \rightarrow H_x = -n_{\text{eff}}\sqrt{\frac{\epsilon_0}{\mu_0}}E_y$$

$$\frac{\partial E_y}{\partial \tilde{x}} = -j\sqrt{\frac{\mu_0}{\epsilon_0}}H_z \rightarrow H_z = j\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{\partial E_y}{\partial \tilde{x}}$$

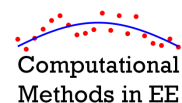
These are substituted into the first equation to get a single equation containing only E_y . This is why it was called the E_y mode.

$$\begin{aligned}
 & -jn_{\text{eff}}H_x - \frac{\partial H_z}{\partial \tilde{x}} = j\sqrt{\frac{\epsilon_0}{\mu_0}}\epsilon_r E_y \\
 & -jn_{\text{eff}}\left(-n_{\text{eff}}\sqrt{\frac{\epsilon_0}{\mu_0}}E_y\right) - \frac{\partial}{\partial \tilde{x}}\left(j\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{\partial E_y}{\partial \tilde{x}}\right) = j\sqrt{\frac{\epsilon_0}{\mu_0}}\epsilon_r E_y \\
 & n_{\text{eff}}^2 E_y - \frac{\partial^2 E_y}{\partial \tilde{x}^2} = \epsilon_r E_y \longrightarrow \frac{\partial^2 E_y}{\partial \tilde{x}^2} + \epsilon_r E_y = n_{\text{eff}}^2 E_y
 \end{aligned}$$

Slab Waveguide Analysis

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Eigen-Value Problem



For optical problems, we like to put everything in terms of refractive index. We have $\epsilon_r = n^2$.

$$\frac{\partial^2 E_y}{\partial \tilde{x}^2} + n^2 E_y = n_{\text{eff}}^2 E_y$$

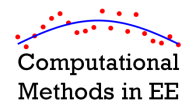
Our governing equation can be rearranged to the form of a standard eigen-value problem.

$$\left[\frac{\partial^2}{\partial \tilde{x}^2} + n^2(x) \right] E_y(x) = n_{\text{eff}}^2 E_y(x)$$

Slab Waveguide Analysis

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Matrix Form



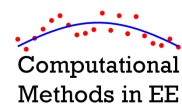
We go term-by-term to write the equation in matrix form.

$$\left[\frac{\partial^2}{\partial \tilde{x}^2} + n^2(x) \right] E_y(x) = n_{\text{eff}}^2 E_y(x)$$

$$(\mathbf{D}_{\tilde{x}}^2 + \mathbf{n}^2) \mathbf{e}_y = n_{\text{eff}}^2 \mathbf{e}_y$$

Slab Waveguide Analysis

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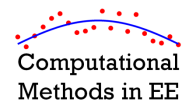


Implementation

Slab Waveguide Analysis

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Implementation Outline

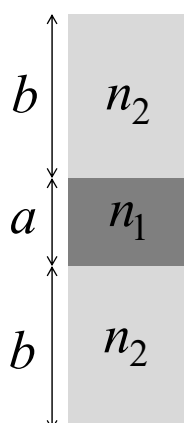
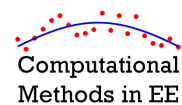


1. Initialize MATLAB
2. Dashboard (materials, dimensions, etc.)
3. Calculate Grid
4. Build Device on Grid
5. Perform Finite-Difference Analysis
6. Visualize the Results

Slab Waveguide Analysis

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Dashboard



How big should we make b ?
 → Enough to allow the mode to decay to zero before reaching the boundary.

What grid resolution should we use?
 → Convergence

```

% slabdemo.m

% INITIALIZE MATLAB
close all;
clc;
clear all;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% DASHBOARD
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% FREE SPACE WAVELENGTH
lam0 = 1.0;

% SLAB PARAMETERS
n1 = 2.0;
n2 = 1.0;
a = 3*lam0;

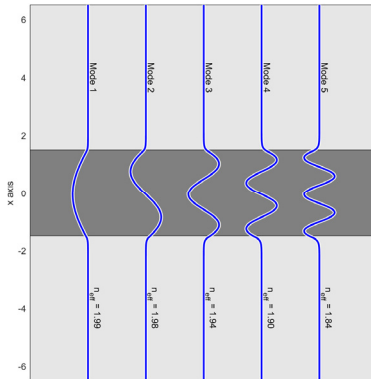
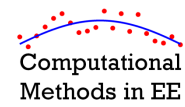
% GRID
b = 5*lam0;
dx = lam0/20;

% NUMBER OF MODES TO CALCULATE
M = 5;
  
```

Slab Waveguide Analysis

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Visualize the Results



```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% VISUALIZE
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% SORT MODES
[~,ind] = sort(real(NEFF),'descend');
V       = V(:,ind);
NEFF    = NEFF(ind);

% OPEN FIGURE WINDOW
figure('Color','w');
hold on;

% DRAW SLAB WAVEGUIDE
x = [0 2*(M+1) 2*(M+1) 0 0];
y = [-b-a/2 -b-a/2 b+a/2 b+a/2 -b-a/2 ];
fill(x,y,0.9*[1 1 1]);
y = [-a/2 -a/2 a/2 a/2 -a/2 ];
fill(x,y,0.5*[1 1 1]);

% DRAW AND LABEL MODES
for m = 1 : M
    x0 = 2*m;
    y0 = (a + b)/2;
    x = x0 + 3*v(i,m);
    y = linspace(-b-a/2,b+a/2,Nx);
    line(x,y,'Color','w','LineWidth',4);
    h = line(x,y,'Color','b','LineWidth',2);
    text(x0,y0,['Mode ' num2str(m)],'Rotation',-90,...
        'HorizontalAlignment','center','VerticalAlignment','bottom');
    text(x0,y0,['n_eff = ' num2str(NEFF(m),'%.2f')],'Rotation',-90,...
        'HorizontalAlignment','center','VerticalAlignment','bottom');
end

% SET GRAPHICS VIEW
hold off;
h2 = get(h,'Parent');
set(h2,'XTick',[]);
axis equal tight;
ylabel('x axis');

```

Slab Waveguide Analysis

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