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EE 5337
Computational Electromagnetics (CEM)

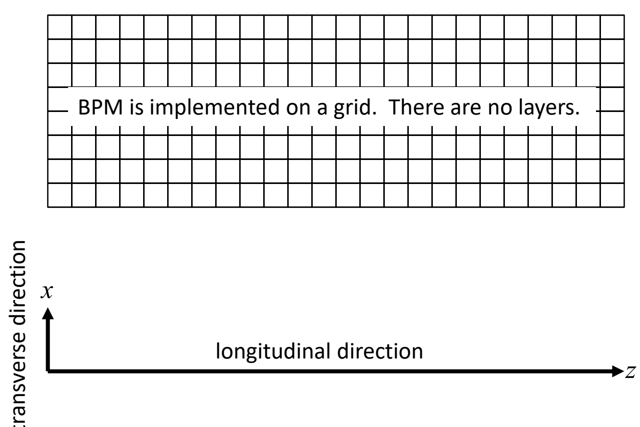
Lecture #17
Beam Propagation Method

Outline

- Overview
- Formulation of 2D finite-difference beam propagation method (FD-BPM)
- Implementation of 2D FD-BPM
- Formulation of 3D FD-BPM
- Alternative formulations of BPM
  - FFT-BPM
  - Wide Angle FD-BPM
  - Bi-Directional BPM

Overview

Geometry of BPM

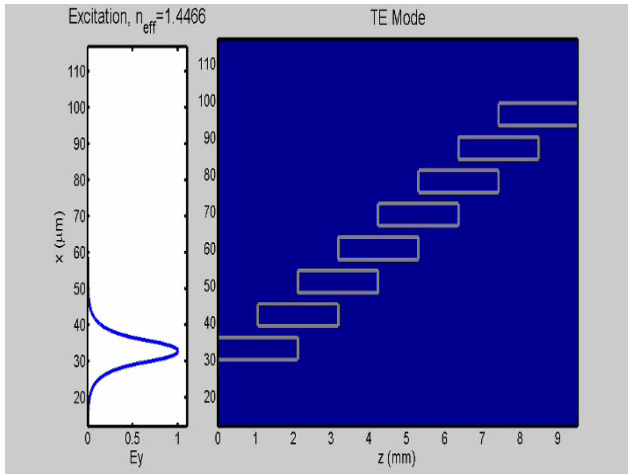


BPM is primarily a “forward” propagating algorithm where the dominant direction of propagation is longitudinal.

The grid is computed and interpreted as it is in FDFD. The algorithm and implementation looks more like the method of lines than it does FDFD.

Example Simulation of a Coupled-Line Filter

This animation is NOT of the wave propagating through the device. Instead, it is the sequence of how the solution is calculated.



Formulation of 2D Finite-Difference Beam Propagation Method

Starting Point

We start with Maxwell's equations in the following form.

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu'_{xx}\tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon'_{xx}E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu'_{yy}\tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon'_{yy}E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu'_{zz}\tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon'_{zz}E_z$$

Recall that we have normalized the grid according to

$$x' = k_0 x \qquad y' = k_0 y \qquad z' = k_0 z$$

Recall that the material properties potentially incorporate a PML at the x and y axis boundaries (propagation along z).

$$\mu'_{xx} = \mu_{xx}\frac{s_y}{s_x} \qquad \mu'_{yy} = \mu_{yy}\frac{s_x}{s_y} \qquad \mu'_{zz} = \mu_{zz}s_x s_y$$

$$\varepsilon'_{xx} = \varepsilon_{xx}\frac{s_y}{s_x} \qquad \varepsilon'_{yy} = \varepsilon_{yy}\frac{s_x}{s_y} \qquad \varepsilon'_{zz} = \varepsilon_{zz}s_x s_y$$

Reduction to Two Dimensions

Assuming the device is uniform along the y direction,

$$\frac{\partial}{\partial y'} = 0$$

and Maxwell's equations reduce to

$$\cancel{\frac{\partial E_z}{\partial y'}} - \frac{\partial E_y}{\partial z'} = \mu'_{xx}\tilde{H}_x \qquad \cancel{\frac{\partial \tilde{H}_z}{\partial y'}} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon'_{xx}E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu'_{yy}\tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon'_{yy}E_y$$

$$\frac{\partial E_y}{\partial x'} - \cancel{\frac{\partial E_x}{\partial y'}} = \mu'_{zz}\tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \cancel{\frac{\partial \tilde{H}_x}{\partial y'}} = \varepsilon'_{zz}E_z$$

$$-\frac{\partial E_y}{\partial z'} = \mu'_{xx}\tilde{H}_x \qquad -\frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon'_{xx}E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu'_{yy}\tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon'_{yy}E_y$$

$$\frac{\partial E_y}{\partial x'} = \mu'_{zz}\tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} = \varepsilon'_{zz}E_z$$

Two Distinct Modes

We see that Maxwell's equations have decoupled into two distinct modes.

<u>E Mode</u>

$$\frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon'_{yy} E_y$$

$$-\frac{\partial E_y}{\partial z'} = \mu'_{xx} \tilde{H}_x$$

$$\frac{\partial E_y}{\partial x'} = \mu'_{zz} \tilde{H}_z$$

<u>H Mode</u>

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu'_{yy} \tilde{H}_y$$

$$-\frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon'_{xx} E_x$$

$$\frac{\partial \tilde{H}_y}{\partial x'} = \varepsilon'_{zz} E_z$$

Slowly Varying Envelope Approximation

Assuming the field is not changing rapidly, we can write the field as

$$\vec{E}(x,z) \approx \vec{\xi}(x,z) e^{jn_{\text{eff}} z'} \qquad \vec{\tilde{H}}(x,z) \approx \vec{\psi}(x,z) e^{jn_{\text{eff}} z'}$$

Not a good approximation to make for metamaterials and photonic crystals!

Better for lenses, waveguides, etc.

Substituting these solutions into our two sets of equations yields

<u>E Mode</u>

$$\frac{\partial}{\partial z'} \psi_x(x,z) e^{jn_{\text{eff}} z'} - \frac{\partial}{\partial x'} \psi_z(x,z) e^{jn_{\text{eff}} z'} = \varepsilon'_{yy} \xi_y(x,z) e^{jn_{\text{eff}} z'}$$

$$-\frac{\partial}{\partial z'} \xi_y(x,z) e^{jn_{\text{eff}} z'} = \mu'_{xx} \psi_x(x,z) e^{jn_{\text{eff}} z'}$$

$$\frac{\partial}{\partial x'} \xi_y(x,z) e^{jn_{\text{eff}} z'} = \mu'_{zz} \psi_z(x,z) e^{jn_{\text{eff}} z'}$$

$$\Downarrow$$

$$jn_{\text{eff}} \psi_x + \frac{\partial \psi_x}{\partial z'} - \frac{\partial \psi_z}{\partial x'} = \varepsilon'_{yy} \xi_y$$

$$-jn_{\text{eff}} \xi_y - \frac{\partial \xi_y}{\partial z'} = \mu'_{xx} \psi_x$$

$$\frac{\partial \xi_y}{\partial x'} = \mu'_{zz} \psi_z$$

<u>H Mode</u>

$$\frac{\partial}{\partial z'} \xi_x(x,z) e^{jn_{\text{eff}} z'} - \frac{\partial}{\partial x'} \xi_z(x,z) e^{jn_{\text{eff}} z'} = \mu'_{yy} \psi_y(x,z) e^{jn_{\text{eff}} z'}$$

$$-\frac{\partial}{\partial z'} \psi_y(x,z) e^{jn_{\text{eff}} z'} = \varepsilon'_{xx} \xi_x(x,z) e^{jn_{\text{eff}} z'}$$

$$\frac{\partial}{\partial x'} \psi_y(x,z) e^{jn_{\text{eff}} z'} = \varepsilon'_{zz} \xi_z(x,z) e^{jn_{\text{eff}} z'}$$

$$\Downarrow$$

$$jn_{\text{eff}} \xi_x + \frac{\partial \xi_x}{\partial z'} - \frac{\partial \xi_z}{\partial x'} = \mu'_{yy} \psi_y$$

$$-jn_{\text{eff}} \psi_y - \frac{\partial \psi_y}{\partial z'} = \varepsilon'_{xx} \xi_x$$

$$\frac{\partial \psi_y}{\partial x'} = \varepsilon'_{zz} \xi_z$$

Matrix Form of Differential Equations

Each of these equations is written once for every point in the grid. This large set of equations can be written in matrix form as

E Mode

$$jn_{\text{eff}}\psi_x + \frac{\partial \psi_x}{\partial z'} - \frac{\partial \psi_z}{\partial x'} = \varepsilon'_{yy}\xi_y$$
$$-jn_{\text{eff}}\xi_y - \frac{\partial \xi_y}{\partial z'} = \mu'_{xx}\psi_x$$
$$\frac{\partial \xi_y}{\partial x'} = \mu'_{zz}\psi_z$$

$$\Downarrow$$

$$jn_{\text{eff}}\mathbf{h}_x + \frac{d\mathbf{h}_x}{dz'} - \mathbf{D}_x^h\mathbf{h}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$
$$-jn_{\text{eff}}\mathbf{e}_y - \frac{d\mathbf{e}_y}{dz'} = \boldsymbol{\mu}_{xx}\mathbf{h}_x$$
$$\mathbf{D}_x^e\mathbf{e}_y = \boldsymbol{\mu}_{zz}\mathbf{h}_z$$

H Mode

$$jn_{\text{eff}}\xi_x + \frac{\partial \xi_x}{\partial z'} - \frac{\partial \xi_z}{\partial x'} = \mu'_{yy}\psi_y$$
$$-jn_{\text{eff}}\psi_y - \frac{\partial \psi_y}{\partial z'} = \varepsilon'_{xx}\xi_x$$
$$\frac{\partial \psi_y}{\partial x'} = \varepsilon'_{zz}\xi_z$$

$$\Downarrow$$

$$jn_{\text{eff}}\mathbf{e}_x + \frac{d\mathbf{e}_x}{dz'} - \mathbf{D}_x^e\mathbf{e}_z = \boldsymbol{\mu}_{yy}\mathbf{h}_y$$
$$-jn_{\text{eff}}\mathbf{h}_y - \frac{d\mathbf{h}_y}{dz'} = \boldsymbol{\varepsilon}_{xx}\mathbf{e}_x$$
$$\mathbf{D}_x^h\mathbf{h}_y = \boldsymbol{\varepsilon}_{zz}\mathbf{e}_z$$

Lecture 17 Slide 11

Wave Equation for E-Mode

We can reduce the set of three equations to a single equation. This is the matrix wave equation.

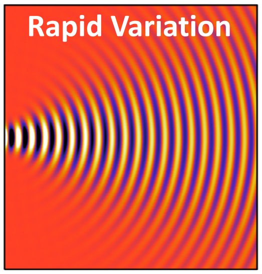
$$jn_{\text{eff}}\mathbf{h}_x + \frac{d\mathbf{h}_x}{dz'} - \mathbf{D}_x^H\mathbf{h}_z = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$
$$-jn_{\text{eff}}\mathbf{e}_y - \frac{d\mathbf{e}_y}{dz'} = \boldsymbol{\mu}_{xx}\mathbf{h}_x \rightarrow \mathbf{h}_x = -\boldsymbol{\mu}_{xx}^{-1}\left(jn_{\text{eff}}\mathbf{I} + \frac{\partial}{\partial z'}\right)\mathbf{e}_y$$
$$\mathbf{D}_x^E\mathbf{e}_y = \boldsymbol{\mu}_{zz}\mathbf{h}_z \rightarrow \mathbf{h}_z = \boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_x^E\mathbf{e}_y$$

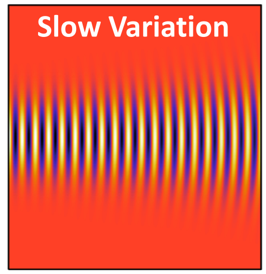
$$-jn_{\text{eff}}\boldsymbol{\mu}_{xx}^{-1}\left(jn_{\text{eff}}\mathbf{I} + \frac{d}{dz'}\right)\mathbf{e}_y - \frac{d}{dz'}\boldsymbol{\mu}_{xx}^{-1}\left(jn_{\text{eff}}\mathbf{I} + \frac{d}{dz'}\right)\mathbf{e}_y - \mathbf{D}_x^H\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_x^E\mathbf{e}_y = \boldsymbol{\varepsilon}_{yy}\mathbf{e}_y$$
$$\downarrow$$
$$\frac{d^2\mathbf{e}_y}{dz'^2} + j2n_{\text{eff}}\frac{d\mathbf{e}_y}{dz'} + \boldsymbol{\mu}_{xx}\mathbf{D}_x^H\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_x^E\mathbf{e}_y + \left(\boldsymbol{\mu}_{xx}\boldsymbol{\varepsilon}_{yy} - n_{\text{eff}}^2\mathbf{I}\right)\mathbf{e}_y = 0$$

Lecture 17 Slide 12

Small Angle Approximation (1 of 2)

In many cases, the envelope of the electromagnetic field does not change rapidly as it propagates.





When this is the case, we can make the "small angle approximation."

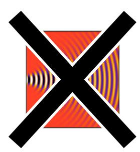
$$\frac{d^2\mathbf{e}_y}{dz'^2} << \frac{d\mathbf{e}_y}{dz'}$$

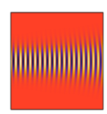
← also called the Fresnel approximation

Small Angle Approximation (2 of 2)

Given the small angle approximation

$$\frac{d^2\mathbf{e}_y}{dz'^2} << \frac{d\mathbf{e}_y}{dz'}$$





We can drop the $\frac{\partial^2\mathbf{e}_y}{\partial z'^2}$ term from our wave equation.

$$\cancel{\frac{d^2\mathbf{e}_y}{dz'^2}} + j2n_{\text{eff}}\frac{d\mathbf{e}_y}{dz'} + \boldsymbol{\mu}_{xx}\mathbf{D}_x^H\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_x^E\mathbf{e}_y + \left(\boldsymbol{\mu}_{xx}\boldsymbol{\varepsilon}_{yy} - n_{\text{eff}}^2\mathbf{I}\right)\mathbf{e}_y = 0$$

$$\downarrow$$

$$j2n_{\text{eff}}\frac{d\mathbf{e}_y}{dz'} + \boldsymbol{\mu}_{xx}\mathbf{D}_x^H\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_x^E\mathbf{e}_y + \left(\boldsymbol{\mu}_{xx}\boldsymbol{\varepsilon}_{yy} - n_{\text{eff}}^2\mathbf{I}\right)\mathbf{e}_y = 0$$

$$\downarrow$$

$$\frac{d\mathbf{e}_y}{dz'} = \frac{j}{2n_{\text{eff}}}\left(\boldsymbol{\mu}_{xx}\mathbf{D}_x^H\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_x^E + \boldsymbol{\mu}_{xx}\boldsymbol{\varepsilon}_{yy} - n_{\text{eff}}^2\mathbf{I}\right)\mathbf{e}_y$$

Different Finite-Difference Solutions

First, we write our matrix wave equation more compactly as

$$\frac{d\mathbf{e}_y}{dz'} = \frac{j}{2n_{\text{eff}}}\mathbf{A}\mathbf{e}_y \qquad \mathbf{A} = \boldsymbol{\mu}_{xx}\mathbf{D}_x^H\boldsymbol{\mu}_{zz}^{-1}\mathbf{D}_x^E + \boldsymbol{\mu}_{xx}\boldsymbol{\varepsilon}_{yy} - n_{\text{eff}}^2\mathbf{I}$$

How do we approximate the z-derivative with a finite-difference?

$$\frac{\mathbf{e}_y^{i+1} - \mathbf{e}_y^i}{\Delta z'} = \frac{j}{2n_{\text{eff}}}\mathbf{A}_i\mathbf{e}_y^i \qquad \text{Forward Euler}$$

$$\frac{\mathbf{e}_y^{i+1} - \mathbf{e}_y^i}{\Delta z'} = \frac{j}{2n_{\text{eff}}}\mathbf{A}_{i+1}\mathbf{e}_y^{i+1} \qquad \text{Backward Euler}$$

Computationally simpler, but solution is nonlinear and only first-order accurate at best. Can be unstable.

$$\frac{\mathbf{e}_y^{i+1} - \mathbf{e}_y^i}{\Delta z'} = \frac{j}{2n_{\text{eff}}}\frac{\mathbf{A}_{i+1}\mathbf{e}_y^{i+1} + \mathbf{A}_i\mathbf{e}_y^i}{2} \qquad \text{Crank-Nicolson}$$

This is a standard finite-difference evaluated at i+0.5

This term is interpolated at i+0.5.

The Crank-Nicolson method is second-order accurate and unconditionally stable.

Lecture 17 Slide 15

Forward Step Equation

Solving our new equation for $\mathbf{e}_y^{i+1}$ leads to

$$\frac{\mathbf{e}_y^{i+1} - \mathbf{e}_y^i}{\Delta z'} = \frac{j}{2n_{\text{eff}}}\frac{\mathbf{A}_{i+1}\mathbf{e}_y^{i+1} + \mathbf{A}_i\mathbf{e}_y^i}{2}$$

$$\downarrow$$

$$\mathbf{e}_y^{i+1} = \left(\mathbf{I} - \frac{j\Delta z'}{4n_{\text{eff}}}\mathbf{A}_{i+1}\right)^{-1}\left(\mathbf{I} + \frac{j\Delta z'}{4n_{\text{eff}}}\mathbf{A}_i\right)\mathbf{e}_y^i$$

We now have a way of calculating the field in a following slice based only the field in the previous slice.

Backward waves are ignored in this formulation.

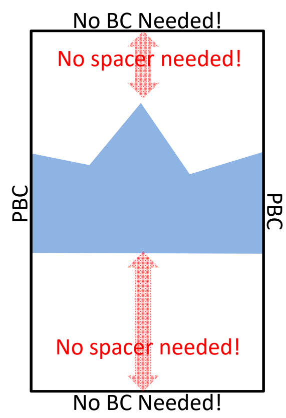
Note: We need to make a good guess for the value of n_{eff}.

Lecture 17 Slide 16

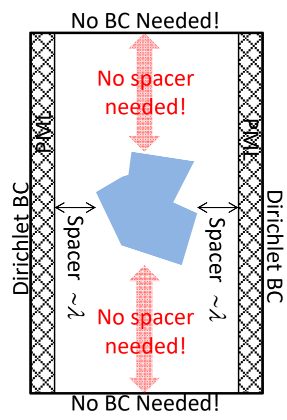
Implementation of 2D FD-BPM

Grid Schemes for BPM

Periodic Structures



Finite Structures



The Effective Refractive Index, n_{eff}

Recall the slowly varying envelop approximation.

$$\vec{E}(x,z) \approx \vec{\xi}(x,z)e^{jn_{\text{eff}}z'} \qquad \vec{H}(x,z) \approx \vec{\psi}(x,z)e^{jn_{\text{eff}}z'}$$

The BPM does not calculate n_{eff}. We must tell BPM what is n_{eff}.

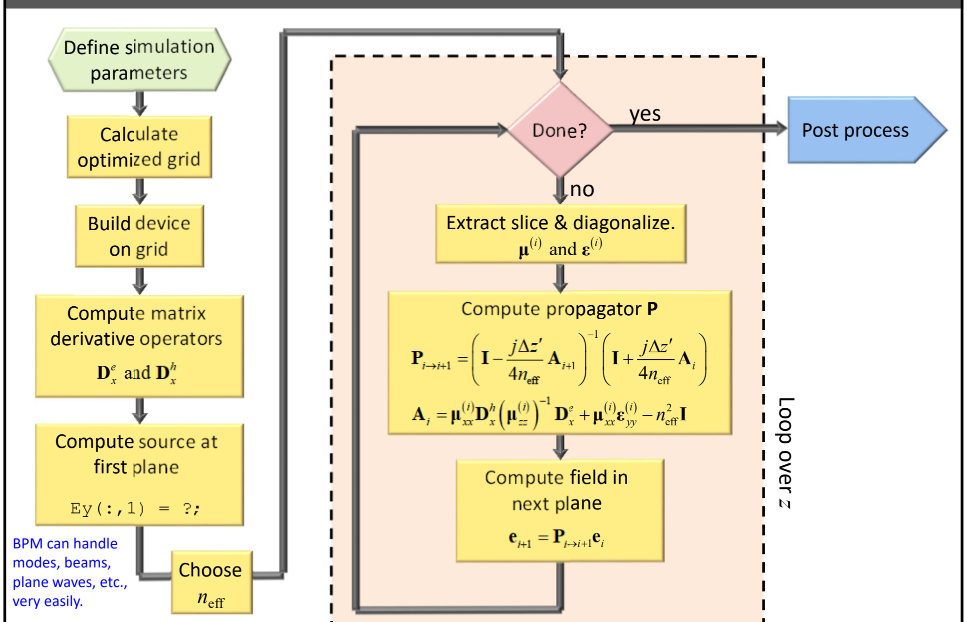
How do we know n_{eff} without modeling the device?

→ We have to calculate it or estimate it.

Techniques:

1. For plane waves and beams, calculate the average refractive index in the cross section of your grid to estimate the longitudinal wave vector.
2. For waveguide problems, calculate the effective index of your guided mode rigorously and use that in BPM.

Block Diagram of BPM



Formulation of 3D Finite-Difference Beam Propagation Method

Starting Point

We have the same starting point as with 2D FD-BPM.

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu'_{xx} \tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon'_{xx} E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu'_{yy} \tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon'_{yy} E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu'_{zz} \tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon'_{zz} E_z$$

Recall that we have normalized the grid according to

$$x' = k_0 x \qquad y' = k_0 y \qquad z' = k_0 z$$

Recall that the material properties potentially incorporate a PML at the x and y axis boundaries (propagation along z).

$$\mu'_{xx} = \mu_{xx} \frac{s_y}{s_x} \qquad \mu'_{yy} = \mu_{yy} \frac{s_x}{s_y} \qquad \mu'_{zz} = \mu_{zz} s_x s_y$$

$$\varepsilon'_{xx} = \varepsilon_{xx} \frac{s_y}{s_x} \qquad \varepsilon'_{yy} = \varepsilon_{yy} \frac{s_x}{s_y} \qquad \varepsilon'_{zz} = \varepsilon_{zz} s_x s_y$$

Slowly Varying Envelope Approximation

Assuming the field is not changing rapidly, we can write the field as

$$\vec{E}(x,z) \approx \vec{\xi}(x,z)e^{jn_{\text{eff}}z'} \qquad \vec{\tilde{H}}(x,z) \approx \vec{\psi}(x,z)e^{jn_{\text{eff}}z'}$$

Not a good approximation to make for metamaterials and photonic crystals!

Good for lenses, waveguides, etc.

Maxwell's equations become

$$\frac{\partial E_z}{\partial y'} - \frac{\partial E_y}{\partial z'} = \mu'_{xx}\tilde{H}_x \qquad \frac{\partial \tilde{H}_z}{\partial y'} - \frac{\partial \tilde{H}_y}{\partial z'} = \varepsilon'_{xx}E_x$$

$$\frac{\partial E_x}{\partial z'} - \frac{\partial E_z}{\partial x'} = \mu'_{yy}\tilde{H}_y \qquad \frac{\partial \tilde{H}_x}{\partial z'} - \frac{\partial \tilde{H}_z}{\partial x'} = \varepsilon'_{yy}E_y$$

$$\frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} = \mu'_{zz}\tilde{H}_z \qquad \frac{\partial \tilde{H}_y}{\partial x'} - \frac{\partial \tilde{H}_x}{\partial y'} = \varepsilon'_{zz}E_z$$

$$\frac{\partial \xi_z}{\partial y'} - \xi_y jn_{\text{eff}} - \frac{\partial \xi_y}{\partial z'} = \mu'_{xx}\psi_x \qquad \frac{\partial \psi_z}{\partial y'} - \psi_y jn_{\text{eff}} - \frac{\partial \psi_y}{\partial z'} = \varepsilon'_{xx}\xi_x$$

$$\xi_x jn_{\text{eff}} + \frac{\partial \xi_x}{\partial z'} - \frac{\partial \xi_z}{\partial x'} = \mu'_{yy}\psi_y \qquad \psi_x jn_{\text{eff}} + \frac{\partial \psi_x}{\partial z'} - \frac{\partial \psi_z}{\partial x'} = \varepsilon'_{yy}\xi_y$$

$$\frac{\partial \xi_y}{\partial x'} - \frac{\partial \xi_x}{\partial y'} = \mu'_{zz}\psi_z \qquad \frac{\partial \psi_y}{\partial x'} - \frac{\partial \psi_x}{\partial y'} = \varepsilon'_{zz}\xi_z$$

Matrix Form of Differential Equations

Each of these equations is written once for every point in the grid. This large set of equations can be written in matrix form as

$$\frac{\partial \xi_z}{\partial y'} - \xi_y jn_{\text{eff}} - \frac{\partial \xi_y}{\partial z'} = \mu'_{xx}\psi_x \qquad \frac{\partial \psi_z}{\partial y'} - \psi_y jn_{\text{eff}} - \frac{\partial \psi_y}{\partial z'} = \varepsilon'_{xx}\xi_x$$

$$\xi_x jn_{\text{eff}} + \frac{\partial \xi_x}{\partial z'} - \frac{\partial \xi_z}{\partial x'} = \mu'_{yy}\psi_y \qquad \psi_x jn_{\text{eff}} + \frac{\partial \psi_x}{\partial z'} - \frac{\partial \psi_z}{\partial x'} = \varepsilon'_{yy}\xi_y$$

$$\frac{\partial \xi_y}{\partial x'} - \frac{\partial \xi_x}{\partial y'} = \mu'_{zz}\psi_z \qquad \frac{\partial \psi_y}{\partial x'} - \frac{\partial \psi_x}{\partial y'} = \varepsilon'_{zz}\xi_z$$

$$\mathbf{D}^e_{y'}\mathbf{e}_z - jn_{\text{eff}}\mathbf{e}_y - \frac{d\mathbf{e}_y}{dz'} = \boldsymbol{\mu}'_{xx}\mathbf{h}_x \qquad \mathbf{D}^h_{y'}\mathbf{h}_z - jn_{\text{eff}}\mathbf{h}_y - \frac{d\mathbf{h}_y}{dz'} = \boldsymbol{\varepsilon}'_{xx}\mathbf{e}_x$$

$$jn_{\text{eff}}\mathbf{e}_x + \frac{d\mathbf{e}_x}{dz'} - \mathbf{D}^e_{x'}\mathbf{e}_z = \boldsymbol{\mu}'_{yy}\mathbf{h}_y \qquad jn_{\text{eff}}\mathbf{h}_x + \frac{d\mathbf{h}_x}{dz'} - \mathbf{D}^h_{x'}\mathbf{h}_z = \boldsymbol{\varepsilon}'_{yy}\mathbf{e}_y$$

$$\mathbf{D}^e_{x'}\mathbf{e}_y - \mathbf{D}^e_{y'}\mathbf{e}_x = \boldsymbol{\mu}'_{zz}\mathbf{h}_z \qquad \mathbf{D}^h_{x'}\mathbf{h}_y - \mathbf{D}^h_{y'}\mathbf{h}_x = \boldsymbol{\varepsilon}'_{zz}\mathbf{e}_z$$

Eliminate Longitudinal Components

We solve the third equation in each set for the longitudinal components $\mathbf{e}_z$ and $\mathbf{h}_z$.

$$\mathbf{h}_z = \boldsymbol{\mu}'^{-1}_{zz}\left(\mathbf{D}^e_{x'}\mathbf{e}_y - \mathbf{D}^e_{y'}\mathbf{e}_x\right) \qquad \mathbf{e}_z = \boldsymbol{\varepsilon}'^{-1}_{zz}\left(\mathbf{D}^h_{x'}\mathbf{h}_y - \mathbf{D}^h_{y'}\mathbf{h}_x\right)$$

We now substitute these expressions into the remaining equations.

$$\mathbf{D}^e_{y'}\boldsymbol{\varepsilon}'^{-1}_{zz}\left(\mathbf{D}^h_{x'}\mathbf{h}_y - \mathbf{D}^h_{y'}\mathbf{h}_x\right) - jn_{\text{eff}}\mathbf{e}_y - \frac{d\mathbf{e}_y}{dz'} = \boldsymbol{\mu}'_{xx}\mathbf{h}_x$$

$$jn_{\text{eff}}\mathbf{e}_x + \frac{d\mathbf{e}_x}{dz'} - \mathbf{D}^e_{x'}\boldsymbol{\varepsilon}'^{-1}_{zz}\left(\mathbf{D}^h_{x'}\mathbf{h}_y - \mathbf{D}^h_{y'}\mathbf{h}_x\right) = \boldsymbol{\mu}'_{yy}\mathbf{h}_y$$

$$\mathbf{D}^h_{y'}\boldsymbol{\mu}'^{-1}_{zz}\left(\mathbf{D}^e_{x'}\mathbf{e}_y - \mathbf{D}^e_{y'}\mathbf{e}_x\right) - jn_{\text{eff}}\mathbf{h}_y - \frac{d\mathbf{h}_y}{dz'} = \boldsymbol{\varepsilon}'_{xx}\mathbf{e}_x$$

$$jn_{\text{eff}}\mathbf{h}_x + \frac{d\mathbf{h}_x}{dz'} - \mathbf{D}^h_{x'}\boldsymbol{\mu}'^{-1}_{zz}\left(\mathbf{D}^e_{x'}\mathbf{e}_y - \mathbf{D}^e_{y'}\mathbf{e}_x\right) = \boldsymbol{\varepsilon}'_{yy}\mathbf{e}_y$$

Rearrange Terms

We rearrange or remaining finite-difference equations and collect the common terms.

$$\frac{d\mathbf{e}_x}{dz'} = -jn_{\text{eff}}\mathbf{e}_x - \mathbf{D}^e_{x'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{y'}\mathbf{h}_x + \left(\boldsymbol{\mu}'_{yy} + \mathbf{D}^e_{x'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{x'}\right)\mathbf{h}_y$$

$$\frac{d\mathbf{e}_y}{dz'} = -jn_{\text{eff}}\mathbf{e}_y - \left(\boldsymbol{\mu}'_{xx} + \mathbf{D}^e_{y'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{y'}\right)\mathbf{h}_x + \mathbf{D}^e_{y'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{x'}\mathbf{h}_y$$

$$\frac{d\mathbf{h}_x}{dz'} = -\mathbf{D}^h_{x'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{y'}\mathbf{e}_x + \left(\boldsymbol{\varepsilon}'_{yy} + \mathbf{D}^h_{x'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{x'}\right)\mathbf{e}_y - jn_{\text{eff}}\mathbf{h}_x$$

$$\frac{d\mathbf{h}_y}{dz'} = -\left(\boldsymbol{\varepsilon}'_{xx} + \mathbf{D}^h_{y'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{y'}\right)\mathbf{e}_x + \mathbf{D}^h_{y'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{x'}\mathbf{e}_y - jn_{\text{eff}}\mathbf{h}_y$$

Block Matrix Form

We can now cast these four matrix equations into two block matrix equations.

$$\frac{d\mathbf{e}_x}{dz'} = -jn_{\text{eff}}\mathbf{e}_x - \mathbf{D}^e_{x'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{y'}\mathbf{h}_x + \left(\boldsymbol{\mu}'_{yy} + \mathbf{D}^e_{x'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{x'}\right)\mathbf{h}_y$$

$$\frac{d\mathbf{e}_y}{dz'} = -jn_{\text{eff}}\mathbf{e}_y - \left(\boldsymbol{\mu}'_{xx} + \mathbf{D}^e_{y'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{y'}\right)\mathbf{h}_x + \mathbf{D}^e_{y'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{x'}\mathbf{h}_y$$

$$\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = -jn_{\text{eff}}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} + \mathbf{P}\begin{bmatrix}\mathbf{h}_x\\ \mathbf{h}_y\end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} -\mathbf{D}^e_{x'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{y'} & \boldsymbol{\mu}'_{yy} + \mathbf{D}^e_{x'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{x'} \\ -\left(\boldsymbol{\mu}'_{xx} + \mathbf{D}^e_{y'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{y'}\right) & \mathbf{D}^e_{y'}\boldsymbol{\varepsilon}'^{-1}_{zz}\mathbf{D}^h_{x'} \end{bmatrix}$$

$$\frac{d\mathbf{h}_x}{dz'} = -\mathbf{D}^h_{x'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{y'}\mathbf{e}_x + \left(\boldsymbol{\varepsilon}'_{yy} + \mathbf{D}^h_{x'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{x'}\right)\mathbf{e}_y - jn_{\text{eff}}\mathbf{h}_x$$

$$\frac{d\mathbf{h}_y}{dz'} = -\left(\boldsymbol{\varepsilon}'_{xx} + \mathbf{D}^h_{y'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{y'}\right)\mathbf{e}_x + \mathbf{D}^h_{y'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{x'}\mathbf{e}_y - jn_{\text{eff}}\mathbf{h}_y$$

$$\frac{d}{dz'}\begin{bmatrix}\mathbf{h}_x\\ \mathbf{h}_y\end{bmatrix} = -jn_{\text{eff}}\begin{bmatrix}\mathbf{h}_x\\ \mathbf{h}_y\end{bmatrix} + \mathbf{Q}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} -\mathbf{D}^h_{x'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{y'} & \boldsymbol{\varepsilon}'_{yy} + \mathbf{D}^h_{x'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{x'} \\ -\left(\boldsymbol{\varepsilon}'_{xx} + \mathbf{D}^h_{y'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{y'}\right) & \mathbf{D}^h_{y'}\boldsymbol{\mu}'^{-1}_{zz}\mathbf{D}^e_{x'} \end{bmatrix}$$

Lecture 17 Slide 27

Matrix Wave Equation

We derive the matrix wave equation by combining the two block matrix equations. First, we solve the first block matrix wave equation for the magnetic field term.

$$\begin{bmatrix}\mathbf{h}_x\\ \mathbf{h}_y\end{bmatrix} = \mathbf{P}^{-1}\left(\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} + jn_{\text{eff}}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}\right)$$

Second, we substitute this into the second block matrix equation.

$$\frac{d}{dz'}\mathbf{P}^{-1}\left(\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} + jn_{\text{eff}}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}\right) = -jn_{\text{eff}}\mathbf{P}^{-1}\left(\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} + jn_{\text{eff}}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}\right) + \mathbf{Q}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

$$\frac{d^2}{dz'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} + jn_{\text{eff}}\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = -jn_{\text{eff}}\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} + n^2_{\text{eff}}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} + \mathbf{PQ}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

$$\frac{d^2}{dz'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} + j2n_{\text{eff}}\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \left(n^2_{\text{eff}}\mathbf{I} + \mathbf{PQ}\right)\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

Lecture 17 Slide 28

Small Angle Approximation

Assuming the fields to not diverge rapidly, then the variation in the field longitudinally will be slow. This means

$$\frac{d^2}{dz'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} << \frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

← also called the Fresnel approximation

This means we can drop the $\frac{\partial^2}{\partial z'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$ term from our wave equation.

$$\cancel{\frac{d^2}{dz'^2}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}} + j2n_{\text{eff}}\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \left(n_{\text{eff}}^2\mathbf{I}+\mathbf{PQ}\right)\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

$$\downarrow$$

$$\frac{d}{dz'}\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} = \frac{1}{j2n_{\text{eff}}}\left(n_{\text{eff}}^2\mathbf{I}+\mathbf{PQ}\right)\begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix}$$

Explicit Finite-Difference Approximation

First, we write our matrix wave equation more compactly as

$$\frac{d\vec{\mathbf{e}}}{dz'} = \frac{1}{j2n_{\text{eff}}}\mathbf{A}\vec{\mathbf{e}} \qquad \vec{\mathbf{e}} = \begin{bmatrix}\mathbf{e}_x\\ \mathbf{e}_y\end{bmatrix} \qquad \mathbf{A} = n_{\text{eff}}^2\mathbf{I}+\mathbf{PQ}$$

We explicitly approximate the z-derivative with a finite-difference.

$$\frac{d\vec{\mathbf{e}}}{dz'} = \frac{1}{j2n_{\text{eff}}}\mathbf{A}\vec{\mathbf{e}}$$

$$\downarrow$$

$$\frac{\vec{\mathbf{e}}_{i+1}-\vec{\mathbf{e}}_i}{\Delta z'} = \frac{1}{j2n_{\text{eff}}}\frac{\mathbf{A}_{i+1}\vec{\mathbf{e}}_{i+1}+\mathbf{A}_i\vec{\mathbf{e}}_i}{2} \qquad \vec{\mathbf{e}}_i = \begin{bmatrix}\mathbf{e}_{x,i}\\ \mathbf{e}_{y,i}\end{bmatrix} \qquad \mathbf{A}_i = n_{\text{eff},i}^2\mathbf{I}+\mathbf{P}_i\mathbf{Q}_i$$

This is a standard finite-difference evaluated at i+0.5

This term is interpolated at i+0.5.

Note: This is called the Crank-Nicolson scheme because it is a central finite-difference.

Forward Step Equation

Solving our new equation for $\vec{\mathbf{e}}_{i+1}$ leads to

$$\frac{\vec{\mathbf{e}}_{i+1}-\vec{\mathbf{e}}_i}{\Delta z'}=\frac{1}{j2n_{\text{eff}}}\frac{\mathbf{A}_{i+1}\vec{\mathbf{e}}_{i+1}+\mathbf{A}_i\vec{\mathbf{e}}_i}{2}$$
$$\downarrow$$
$$\vec{\mathbf{e}}_{i+1}=\left(\mathbf{I}-\frac{\Delta z'}{j4n_{\text{eff}}}\mathbf{A}_{i+1}\right)^{-1}\left(\mathbf{I}+\frac{\Delta z'}{j4n_{\text{eff}}}\mathbf{A}_i\right)\vec{\mathbf{e}}_i$$

We now have a way of calculating the field in a following slice based only the field in the previous slice.

Backward waves are ignored in this formulation.

Note: We need to make a good guess for the value of n_{eff}.

Lecture 17 Slide 31

Alternative Formulations of BPM

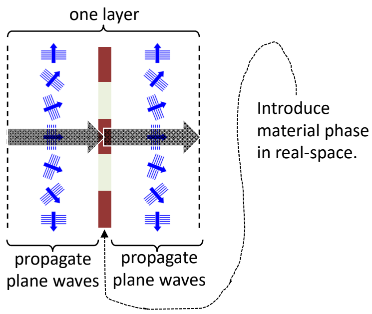
Lecture 17 Slide 32

FFT-BPM

The FFT based BPM was the first BPM. It was essentially replaced by FD-BPM because FFT-BPM has the following disadvantages:

- Simulations were slow (FFTs are computationally intensive)
- Discretization in the transverse dimension must be uniform
- No transparent boundary condition could be used
- Very small discretization widths were not feasible
- Polarization cannot be treated
- Inaccurate for high contrast devices
- Propagation step must be small

M. D. Feit, J. A. Fleck, Jr., "Light propagation in graded-index optical fibers," Applied Optics, Vol. 17, No. 24, December 1978.



Algorithm to Propagate One Layer

1. FFT the fields to calculate plane wave spectrum.
2. Add phase for one half of layer to plane waves according to their longitudinal wave vector.
3. Inverse FFT at mid-point to reconstruct the real-space field.
4. Introduce the phase due to the materials in the layer.
5. Repeat steps 1 to 3 for the second half of the layer.

Wide Angle FD-BPM – Recurrence Formula

The wave equation before the small angle approximation was made was:

$$\frac{\partial^2 \xi_y}{\partial z'^2} + j2n_{\text{eff}}\frac{\partial \xi_y}{\partial z'} + \mu'_{xx}\frac{\partial}{\partial x'}\frac{1}{\mu'_{zz}}\frac{\partial \xi_y}{\partial x'} + \left(\mu'_{xx}\varepsilon'_{yy} - n_{\text{eff}}^2\right)\xi_y = 0$$

This can be written more compactly as

$$\frac{\partial^2 \xi_y}{\partial z'^2} + j2n_{\text{eff}}\frac{\partial \xi_y}{\partial z'} + A\xi_y = 0 \qquad A = \mu'_{xx}\frac{\partial}{\partial x'}\frac{1}{\mu'_{zz}}\frac{\partial}{\partial x'} + \mu'_{xx}\varepsilon'_{yy} - n_{\text{eff}}^2$$

We can rearrange this differential equation to derive a recurrence formula for the derivatives.

$$\left[\frac{\partial}{\partial z'}\right]\xi_y = \left[-\frac{\dfrac{A}{j2n_{\text{eff}}}}{1+\dfrac{1}{j2n_{\text{eff}}}\dfrac{\partial}{\partial z'}}\right]\xi_y \quad \Rightarrow \quad \boxed{\left.\frac{\partial}{\partial z'}\right|_m = \frac{-\dfrac{A}{j2n_{\text{eff}}}}{1+\dfrac{1}{j2n_{\text{eff}}}\left.\dfrac{\partial}{\partial z'}\right|_{m-1}}}$$

Wide Angle FD-BPM – Padé Approximant Operators

We initialize the recurrence with

$$\left.\frac{\partial}{\partial z'}\right|_{-1} = 0$$

0th Order (Small Angle)

$$\left.\frac{\partial}{\partial z'}\right|_{0} = \frac{-\dfrac{A}{j2n_{\text{eff}}}}{1+\dfrac{1}{j2n_{\text{eff}}}\left.\dfrac{\partial}{\partial z'}\right|_{-1}} = j\frac{A}{2n_{\text{eff}}}$$

1st Order (Wide Angle)

$$\left.\frac{\partial}{\partial z'}\right|_{1} = \frac{-\dfrac{A}{j2n_{\text{eff}}}}{1+\dfrac{1}{j2n_{\text{eff}}}\left.\dfrac{\partial}{\partial z'}\right|_{0}} = j\frac{\dfrac{A}{2n_{\text{eff}}}}{1+\dfrac{A}{4n_{\text{eff}}^2}}$$

2nd Order (Wide Angle)

$$\left.\frac{\partial}{\partial z'}\right|_{2} = \frac{-\dfrac{A}{j2n_{\text{eff}}}}{1+\dfrac{1}{j2n_{\text{eff}}}\left.\dfrac{\partial}{\partial z'}\right|_{1}} = j\frac{\dfrac{A}{2n_{\text{eff}}}+\dfrac{A^2}{8n_{\text{eff}}^3}}{1+\dfrac{A}{2n_{\text{eff}}^2}}$$

The numerator and denominator are polynomials of the operator A of orders N and D respectively.

This leads to the Padé Approximate operators denoted as Padé(N, D)

Wide Angle FD-BPM – Implementation

Previously we solved $\frac{\partial^2 \xi_y}{\partial z'^2} + j2n_{\text{eff}}\frac{\partial \xi_y}{\partial z'} + A\xi_y = 0$ by setting $\frac{\partial^2 \xi_y}{\partial z'^2} = 0$. This was the small angle approximation.

For wide angle BPM, we instead solve $\frac{\partial \xi_y}{\partial z'} = -j\frac{N(A)}{D(A)}\xi_y$ where $N(A)$ and $D(A)$ are the polynomials of A.

This equation is usually implemented with finite-differences using the "multi-step" method.

G. Ronald Hadley, "Wide-angle beam propagation using Padé approximant operators," Optics Lett., Vol. 17, No. 20, October 1992.

Bi-Directional BPM

The beam propagation method inherently propagates waves in only the forward direction.

It is possible to modify the method so as to account for backward scattered waves.

This is accomplished in a manner similar to how we derived scattering matrices.

By the time BPM is modified to be bidirectional and wide-angle, it approaches being a rigorous method. The implementation, however, is tedious. At this point, **use the method of lines** which is fully rigorous and has a simpler implementation.

Hatem El-Refaei, David Yevick, and Ian Betty, "Stable and Noniterative Bidirectional Beam Propagation Method," IEEE Photonics Technol. Lett, Vol. 12, No. 4, April 2000.