Preliminary Topics

Outline

• Review of Linear Algebra
• Wave Vectors
• Wave Polarization
• Index Ellipsoids
• Electromagnetic Behavior at an Interface
  – Phase matching at an interface
  – The Fresnel equations
  – Visualization
• Image Theory
Review of Linear Algebra

Matrices Represent Sets of Equations

A set of linear algebraic equations can be written in “matrix” form.

\[
\begin{align*}
a_{11}w + a_{12}x + a_{13}y + a_{14}z &= b_1 \\
a_{21}w + a_{22}x + a_{23}y + a_{24}z &= b_2 \\
a_{31}w + a_{32}x + a_{33}y + a_{34}z &= b_3 \\
a_{41}w + a_{42}x + a_{43}y + a_{44}z &= b_4
\end{align*}
\]

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
w \\
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\]
**Interpretation of Matrices**

\[ a_{11}x + a_{12}y + a_{13}z = b_1 \]
\[ a_{21}x + a_{22}y + a_{23}z = b_2 \]
\[ a_{31}x + a_{32}y + a_{33}z = b_3 \]

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix} =
\begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3
\end{bmatrix}
\]

From a purely mathematical perspective, this interpretation does not make sense. This interpretation will be highly useful and insightful because of how we derive the equations.

**Compact Matrix Notation**

Matrices and vectors can be represented and treated as single variables.

\[
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34} \\
    a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
    w \\
    x \\
    y \\
    z
\end{bmatrix} =
\begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    b_4
\end{bmatrix}
\]

\[ A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24} \\
    a_{31} & a_{32} & a_{33} & a_{34} \\
    a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix} \quad \begin{bmatrix}
    w \\
    x \\
    y \\
    z
\end{bmatrix} = \begin{bmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    b_4
\end{bmatrix} \quad \text{or} \quad [A][x]=[b] \]

Ax = b
Matrices Require Special Algebra

Commutative Laws

\[ AB \neq BA \]

\[
\begin{align*}
AB &= BA \\
A + B &= B + A
\end{align*}
\]

Associative Laws

\[
\begin{align*}
(AB)C &= A(BC) \\
A(B + C) &= AB + AC
\end{align*}
\]

\[
\begin{align*}
(A + B)C &= A + (B + C) \\
(A + B)C &= AC + BC
\end{align*}
\]

Distributive Laws

\[
\begin{align*}
\alpha(A + B) &= \alpha A + \alpha B \\
\alpha(AB) &= (\alpha A)B = A(\alpha B)
\end{align*}
\]

\[
\alpha I + A = \begin{bmatrix}
\alpha + a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & \alpha + a_{nn}
\end{bmatrix}
\]

Multiplication with a Scalar

\[
(A^{-1})^T = A \\
(AB)^T = B^T A^T
\]

Addition with a Scalar

\[
\begin{align*}
\alpha I + A = & \begin{bmatrix}
\alpha + a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & \alpha + a_{nn}
\end{bmatrix} \\
& = \begin{bmatrix}
\alpha I + A = & \begin{bmatrix}
\alpha + a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & \alpha + a_{nn}
\end{bmatrix}
\end{align*}
\]

Matrix Inverses and Transposes

\[
\begin{align*}
(A^{-1})^T &= A \\
(AB)^T &= B^T A^T
\end{align*}
\]

Manipulating Matrix Equations

Example #1: Dividing both sides by a matrix on the right

\[
\begin{align*}
(A + B) C &= D + E \\
(A + B) C^{-1} &= (D + E) C^{-1}
\end{align*}
\]

Starting equation

Post-multiply both sides by \( C^{-1} \)

Recall that \( CC^{-1} = I \)

Example #2: Dividing both sides by a matrix on the left

\[
\begin{align*}
C(A + B) &= D + E \\
C^{-1} C(A + B) &= C^{-1} (D + E)
\end{align*}
\]

Starting equation

Pre-multiply both sides by \( C^{-1} \)

Recall that \( C^{-1} C = I \)

Example #3: Simplify an expression

\[
\begin{align*}
(C^T A)^{-1} + D &= BC + D \\
(C^T A)^{-1} &= BC
\end{align*}
\]

Starting equation

Subtract \( D \) from both sides

Recall inverse of a product rule

\[
\begin{align*}
A^{-1} C &= BC \\
A^{-1} C^{-1} &= BC^{-1}
\end{align*}
\]

Recall inverse of a product rule

Post-multiply both sides by \( C^{-1} \)

Recall that \( C^{-1} C = I \)
The Zero and the Identity Matrices

Zero Matrix

\[
0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}
\]

\[
0 \cdot A = A \cdot 0 = 0 \\
0 + A = A + 0 = A \\
A - A = 0
\]

Identity Matrix

\[
I = \begin{bmatrix} 1 & \ddots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}
\]

\[
I \cdot A = A \cdot I = A \\
A^{-1}A = AA^{-1} = I
\]

Matrix Division

It is very rare to calculate the inverse of a matrix because it is very computationally intensive to do this.

At first glance, matrix division appears to require calculating the inverse of a matrix, but highly efficient algorithms exist that do not need to do this.

\[
\text{Pre-Division: } A = B^{-1}C \\
\text{A = inv}(B) \ast C; \\
A = B \backslash C;
\]

\[
\text{Post-Division: } A = CB^{-1} \\
A = \text{inv}(B) \ast C; \\
A = C / B;
\]

Despite that both of these equations are dividing by \(B\), pre- and post-division do NOT lead to the same result.

\[
B^{-1}C \neq CB^{-1}
\]
Matrix Division With More than Two Terms

Suppose we must evaluate the following matrix expression.

\[ A = BC^{-1}D \]

Would the following MATLAB code work?

\[ A = B*C\backslash D; \]

No! Remember the order of operations. MATLAB will first multiply \( B \) and \( C \).

\[ A = BC \]

It will then backward divide on \( D \).

\[ A = (BC)^{-1}D \]

So what is the correct code?

\[ A = B/C*D; \]
\[ A = B*(C\backslash D); \]

It might be a good idea to time these two calculations to see which is faster.

Proper Notation for Matrix Division

It is almost never correct to write matrix division as a fraction.

\[ \frac{A}{B} \]

Why?

Does this represent predivision or postdivision?

\[ B^{-1}A \text{ or } AB^{-1} \]

It is impossible to say, thus fraction notation is incorrect for matrices.

One exception \( \Rightarrow \) When both \( A \) and \( B \) are diagonal matrices, both pre- and post-division will give the same answer.

\[
\begin{bmatrix}
    b_{11} & \cdots & b_{1n} \\
    \vdots & \ddots & \vdots \\
    b_{n1} & \cdots & b_{nn}
\end{bmatrix}
\begin{bmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
    h_{11} \\
    \vdots \\
    h_{nn}
\end{bmatrix}
= 
\begin{bmatrix}
    a_{11}/h_{11} & \cdots & a_{1n}/h_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1}/h_{n1} & \cdots & a_{nn}/h_{nn}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    b_{11} & \cdots & b_{1n} \\
    \vdots & \ddots & \vdots \\
    b_{n1} & \cdots & b_{nn}
\end{bmatrix}
\begin{bmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
    h_{11} \\
    \vdots \\
    h_{nn}
\end{bmatrix}
= 
\begin{bmatrix}
    a_{11}/h_{11} & \cdots & a_{1n}/h_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1}/h_{n1} & \cdots & a_{nn}/h_{nn}
\end{bmatrix}
\]
Ax = b Vs. Ax = \lambda x

Standard Linear Problem

Ax = b

- Has only one answer
- Requires a source

Eigen-Value Problem

Ax = \lambda x

- Has an infinite number of answers
- Modes
- No source

Wave Vectors
Wave Vector $\vec{k}$

The wave vector $\vec{k}$ conveys two pieces of information: (1) Magnitude conveys the wavelength $\lambda$ inside the medium, and (2) direction conveys the direction of the wave and is perpendicular to the wave fronts.

$$|\vec{k}| = \frac{2\pi}{\lambda}$$

$$\vec{k} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$$

Magnitude Conveys Wavelength

Most fundamentally, the magnitude of the wave vector conveys the wavelength of the wave inside of the medium.

$$|\vec{k}_1| = \frac{2\pi}{\lambda_1}$$

$$|\vec{k}_2| = \frac{2\pi}{\lambda_2}$$
Magnitude May Convey Refractive Index

When the frequency of a wave is known, the magnitude of the wave vector conveys refractive index.

\[ |\vec{k}_1| = \frac{2\pi n_1}{\lambda_0} = k_0 n_1 \]
\[ |\vec{k}_2| = \frac{2\pi n_2}{\lambda_0} = k_0 n_2 \]
\[ k_0 = \frac{2\pi}{\lambda_0} \]

The Complex Wave Number

A wave travelling the +z direction can be written in terms of the wave number \( k \) as

\[ \vec{E}(z) = \vec{P} e^{-jkz} \]

\[ k = k' - jk'' \]

Substituting this back into the wave solution yields

\[ \vec{E}(z) = \vec{E}_0 e^{-j(k'-jk'')z} = \vec{E}_0 e^{-k'z} e^{-jk''z} \]

attenuation oscillation
**α and β**

A wave travelling the +z direction can also be written in terms of a phase constant $\beta$ and an attenuation coefficient $\alpha$ as

$$\vec{E}(z) = \vec{E}_0 e^{-k'z} e^{-j\beta z}$$

We now have physical meaning to the real and imaginary parts of the wave vector.

$$k = \beta - j\alpha$$

$k' = \text{Re}[k]$ → phase term

$k'' = \text{Im}[k]$ → attenuation term

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi n}{\lambda_0}$$

---

**Waves with Complex $k$**

<table>
<thead>
<tr>
<th>Purely Real $k$</th>
<th>Purely Imaginary $k$</th>
<th>Complex $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform amplitude</td>
<td>Decaying amplitude</td>
<td>Decaying amplitude</td>
</tr>
<tr>
<td>Oscillations move power</td>
<td>No oscillations, no flow of power</td>
<td>Oscillations move power</td>
</tr>
<tr>
<td>Considered to be a propagating wave</td>
<td>Considered to be evanescent</td>
<td>Considered to be a propagating wave (not evanescent)</td>
</tr>
</tbody>
</table>

This implies that these are the only 2.5 configurations that electromagnetic fields can take on.
Wave Polarization
What is Polarization?

Polarization is that property of a radiated electromagnetic wave which describes the time-varying direction and relative magnitude of the electric field vector.

**Linear Polarization (LP)**

**Circular Polarization (CP)**

To determine the handedness of CP, imagine watching the electric field in a plane while the wave is coming at you. Which way does it rotate?

Possibilities for Wave Polarization

Recall that \( \vec{E} \perp \vec{k} \) so the polarization vector \( \vec{P} \) must fall within the plane perpendicular to \( \vec{k} \).

We can decompose the polarization into two orthogonal directions, \( \hat{a} \) and \( \hat{b} \).

\[
\vec{P} = p_a \hat{a} + p_b \hat{b}
\]
Explicit Form to Convey Polarization

Our electromagnetic wave can be now be written as

\[ \mathbf{E} (\mathbf{r}) = \mathbf{F} e^{-j k \cdot \mathbf{r}} = \left( p_a \hat{a} + p_b \hat{b} \right) e^{-j k \cdot \mathbf{r}} \]

\( p_a \) and \( p_b \) are in general complex numbers in order to convey the relative phase of each of these components.

\[ p_a = E_a e^{j \phi_a} \quad p_b = E_b e^{j \phi_b} \]

Substituting \( p_a \) and \( p_b \) into our wave expression gives

\[ \mathbf{E} (\mathbf{r}) = \left[ E_a e^{j \phi_a} \hat{a} + E_b e^{j \phi_b} \hat{b} \right] e^{-j k \cdot \mathbf{r}} = \left[ E_a \hat{a} + E_b e^{j(\phi_b - \phi_a)} \hat{b} \right] e^{j \phi_b} e^{-j k \cdot \mathbf{r}} \]

We interpret \( \phi_b - \phi_a \) as the phase difference between \( p_a \) and \( p_b \).

\[ \delta = \phi_b - \phi_a \]

We interpret \( \phi_a \) as the phase common to both \( p_a \) and \( p_b \).

\[ \theta = \phi_a \]

The final expression is:

\[ \mathbf{E} (\mathbf{r}) = \left( E_a \hat{a} + E_b e^{j \delta} \hat{b} \right) e^{j \theta} e^{-j k \cdot \mathbf{r}} \]

Determining Polarization of a Wave

To determine polarization, it is most convenient to write the expression for the wave that makes polarization explicitly.

\[ \mathbf{E} (\mathbf{r}) = \left( E_a \hat{a} + E_b e^{j \delta} \hat{b} \right) e^{j \theta} e^{-j k \cdot \mathbf{r}} \]

\( E_a = \) amplitude along \( \hat{a} \)
\( E_b = \) amplitude along \( \hat{b} \)
\( \delta = \) phase difference
\( \theta = \) common phase

We can now identify the polarization of the wave...

<table>
<thead>
<tr>
<th>Polarization Designation</th>
<th>Mathematical Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Polarization (LP)</td>
<td>( \delta = 0^\circ )</td>
</tr>
<tr>
<td>Circular Polarization (CP)</td>
<td>( \delta = \pm 90^\circ, E_a = E_b )</td>
</tr>
<tr>
<td>Right-Hand CP (RCP)</td>
<td>( \delta = +90^\circ, E_a = E_b )</td>
</tr>
<tr>
<td>Left-Hand CP (LCP)</td>
<td>( \delta = -90^\circ, E_a = E_b )</td>
</tr>
<tr>
<td>Elliptical Polarization</td>
<td>Everything else</td>
</tr>
</tbody>
</table>
**Handedness Convention**

**As Viewed From Source**
Polarization is taken as the time-varying electric field view with the wave moving away from you. Primarily used in engineering and quantum physics.

**As Viewed From Receiver**
Polarization is taken as the time-varying electric field view with the wave coming toward you. Primarily used in optics and physics.

**Linear Polarization**

A wave travelling in the $+z$ direction is said to be linearly polarized if:

$$\vec{E}(x, y, z) = \vec{P} e^{j k z}$$

$$\vec{P} = (\sin \theta) \hat{x} + (\cos \theta) \hat{y}$$

$\vec{P}$ is called the polarization vector.

For an arbitrary wave,

$$\vec{E}(\vec{r}) = \vec{P} e^{j (k \cdot \vec{r})}$$

$$\vec{P} = (\sin \theta) \hat{a} + (\cos \theta) \hat{b}$$

$$\hat{a} \perp \hat{b} \perp \hat{k}$$

All components of $\vec{P}$ have equal phase.
Linear Polarization

A wave travelling in the $+z$ direction is said to be circularly polarized if:

$$\vec{E}(x, y, z) = \vec{P} e^{j k z} \quad \vec{P} = \hat{x} \pm j \hat{y}$$

$\vec{P}$ is called the polarization vector.

For an arbitrary wave,

$$\vec{E}(\vec{r}) = \vec{P} e^{j k \cdot \vec{r}}$$

$$\vec{P} = \hat{a} \pm j \hat{b}$$

$$\hat{a} \perp \hat{b} \perp \hat{k}$$

The two components of $\vec{P}$ have equal amplitude and are $90^\circ$ out of phase.

Circular Polarization

LCP

RCP

Lecture 3
\[ \text{LP}_x + \text{LP}_y = \text{LP}_{45} \]

A linearly polarized wave can always be decomposed as the sum of two linearly polarized waves that are in phase.

\[ \text{LP}_x + j\text{LP}_y = \text{CP} \]

A circularly polarized wave is the sum of two orthogonal linearly polarized waves that are 90° out of phase.
A linearly polarized wave can be expressed as the sum of a LCP wave and a RCP wave. The phase between the two CP waves determines the tilt of the LP wave.

Why is Polarization Important?

- Different polarizations can behave differently in a device
- Orthogonal polarizations will not interfere with each other
- Polarization becomes critical when analyzing structures that are on the scale of a wavelength
- Focusing properties are different
- Reflection/transmission can be different
- Frequency of resonators
- Cutoff conditions for filters, waveguides, etc.
Poincaré Sphere

The polarization of a wave can be mapped to a unique point on the Poincaré sphere.

Points on opposite sides of the sphere are orthogonal.

See Balanis, Chap. 4.

TE and TM

We use the labels “TE” and “TM” when we are describing the orientation of a linearly polarized wave relative to a device.

**TE/parallel/p** – the electric field is polarized parallel to the plane of incidence.

**TM/parallel/p** – the electric field is polarized parallel to the plane of incidence.
Calculating the Polarization Vectors

Incident Wave Vector

\[ \vec{k}_{\text{inc}} = k_0 n_{\text{inc}} \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} \]

Surface Normal

\[ \hat{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

Unit Vectors of Polarization Directions

\[ \hat{a}_{\text{TE}} = \begin{cases} \hat{y} & \theta = 0^\circ \\ \hat{\vec{n}} \times \vec{k}_{\text{inc}} & \theta \neq 0^\circ \end{cases} \]

\[ \hat{a}_{\text{TM}} = \frac{\hat{a}_{\text{TE}} \times \vec{k}_{\text{inc}}}{|\hat{a}_{\text{TE}} \times \vec{k}_{\text{inc}}|} \]

Composite Polarization Vector

\[ \vec{P} = p_{\text{TE}} \hat{a}_{\text{TE}} + p_{\text{TM}} \hat{a}_{\text{TM}} \]

In CEM, we usually make

\[ |\vec{P}| = 1 \]

Index Ellipsoids
Dispersion Relations

The dispersion relation for a material relates the wave vector to frequency. Essentially, it tells us the refractive index as a function of direction through a material.

It is derived by substituting a plane wave solution into the wave equation.

For a linear, homogeneous, and isotropic (LHI) material, the dispersion relation is:

\[
-k^2 + \frac{k_a^2 + k_b^2 + k_c^2}{n^2} = k_0^2
\]

This can also be written as:

\[
\frac{k_a^2 + k_b^2 + k_c^2}{n^2} - k_0^2 = 0
\]

Index Ellipsoids

From the previous slide, the dispersion relation for a LHI material was:

\[
k_a^2 + k_b^2 + k_c^2 = k_0^2 n^2
\]

This defines a sphere called an “index ellipsoid.”

The vector connecting the origin to a point on the sphere is the \( k \)-vector for that direction. Refractive index \( n \) is calculated from its magnitude.

\[
|\vec{k}| = k_0 n
\]

For LHI materials, the refractive index is the same in all directions.

Think of this as a map of the refractive index as a function of the wave's direction through the medium.
Index Ellipsoids for Uniaxial Materials

Observations
• Both solutions share a common axis.
• This “common” axis looks isotropic with refractive index $n_o$ regardless of polarization.
• Since both solutions share a single axis, these crystals are called “uniaxial.”
• The “common” axis is called:
  - Optic axis
  - Ordinary axis
  - C axis
  - Uniaxial axis
• Deviation from the optic axis will result in two separate possible modes.

$$\left(\frac{k_a^2 + k_b^2 + k_c^2}{n_o^2} - k_0^2\right)\left(\frac{k_a^2 + k_b^2}{n_E^2} - k_0^2\right) = 0$$

Index Ellipsoids for Biaxial Materials

Biaxial materials have all unique refractive indices. Most texts adopt the convention where

$$n_a < n_b < n_c$$

The general dispersion relation cannot be reduced.

Notes and Observations
• The convention $n_a < n_b < n_c$ causes the optic axes to lie in the $a-c$ plane.
• The two solutions can be envisioned as one balloon inside another, pinched together so they touch at only four points.
• Propagation along either of the optic axes looks isotropic, thus the name “biaxial.”
**Direction of Power Flow**

**Isotropic Materials**

\[ \vec{P} \]

\[ \vec{k} \]

Phase propagates in the direction of \( \vec{k} \). Therefore, the refractive index derived from \( |\vec{k}| \) is best described as the phase refractive index. Velocity here is the phase velocity.

Power propagates in the direction of \( \vec{P} \) which is always normal to the surface of the index ellipsoid. From this, we can define a group velocity and a group refractive index.

**Anisotropic Materials**

\[ \vec{P} \]

\[ \vec{k} \]

**Illustration of \( \vec{k} \) versus \( \vec{P} \)**

Negative refraction into an electromagnetic band gap material.

We don’t need a negative refractive index to have negative refraction.
Phase Matching at an Interface

The dispersion relation for isotropic materials is essentially just the Pythagorean theorem. It says a wave sees the same refractive index no matter what direction the wave is travelling.
Index Ellipsoid in Two Different Materials

Material 1 (Low $n$)

$$k_{x,1}^2 + k_{y,1}^2 = |\vec{k}_1|^2 = (k_0 n_1)^2$$

Material 2 (High $n$)

$$k_{x,2}^2 + k_{y,2}^2 = |\vec{k}_2|^2 = (k_0 n_2)^2$$

$n_1 < n_2$

Phase Matching at the Interface Between Two Materials Where $n_1 < n_2$

Material 1

$$k_{x,1}^2 + k_{y,3}^2 = |\vec{k}_1|^2 = (k_0 n_1)^2$$

Material 2

$$k_{x,2}^2 + k_{y,3}^2 = |\vec{k}_2|^2 = (k_0 n_2)^2$$
### Summary of the Phase Matching Trend for $n_1 < n_2$

<table>
<thead>
<tr>
<th>$n_1 &lt; n_2$</th>
<th>Material 1</th>
<th>Material 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{x,1}^2 + k_{y,1}^2 =</td>
<td>\vec{k}_1</td>
<td>^2 = (k_0 n_1)^2$</td>
</tr>
</tbody>
</table>

Properly phased matched at the interface.

### Phase Matching at the Interface Between Two Materials Where $n_1 > n_2$

<table>
<thead>
<tr>
<th>$n_1 &gt; n_2$</th>
<th>Material 1</th>
<th>Material 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{\text{inc}} &lt; \theta_c$</td>
<td>$k_{x,1}^2 + k_{y,1}^2 =</td>
<td>\vec{k}_1</td>
</tr>
<tr>
<td>$\theta_{\text{inc}} &gt; \theta_c$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Summary of the Phase Matching Trend for** \( n_1 > n_2 \)

\[
k^{2,1}_{j,j} + k^{2,2}_{j,j} = |k_j|^2 = (k_0 n_1)^2
\]

Material 1

\[
k^{2,2}_{j,j} + k^{2,2}_{j,j} = |k_j|^2 = (k_0 n_2)^2
\]

Material 2

1. \( \theta_{inc} < \theta_c \)
2. \( \theta_{inc} < \theta_c \)
3. \( \theta_{inc} = \theta_c \)
4. \( \theta_{inc} > \theta_c \)

Properly phased matched at the interface.

**Reflection and Transmission:**

**The Fresnel Equations**
**Reflection, Transmission, and Refraction at an Interface**

**Angles**

\[
\theta_{\text{inc}} = \theta_{\text{ref}} = \theta_1 \\
\sin \theta_1 = n_2 \sin \theta_2
\]

Snell’s Law

**TE Polarization**

\[
r_{\text{TE}} = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_2 \cos \theta_1 + n_1 \cos \theta_2} \\
t_{\text{TE}} = \frac{2n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_2} \\
1 + r_{\text{TE}} = t_{\text{TE}}
\]

**TM Polarization**

\[
r_{\text{TM}} = \frac{n_2 \cos \theta_2 - n_1 \cos \theta_1}{n_2 \cos \theta_2 + n_1 \cos \theta_1} \\
t_{\text{TM}} = \frac{2n_1 \cos \theta_2}{n_2 \cos \theta_2 + n_1 \cos \theta_1} \\
1 + r_{\text{TM}} = t_{\text{TM}}
\]

**Reflectance and Transmittance**

**Reflectance**

The fraction of power \( R \) reflected from an interface is called **reflectance**. It is related to the reflection coefficient \( r \) through

\[
R_{\text{TE}} = \left| r_{\text{TE}} \right|^2 \\
R_{\text{TM}} = \left| r_{\text{TM}} \right|^2 \\
\frac{R_{\text{TE}}}{R_{\text{TM}}} = \left| \frac{r_{\text{TE}}}{r_{\text{TM}}} \right|^2
\]

**Transmittance**

The fraction of power \( T \) transmitted through an interface is called **transmittance**. It is related to the transmission coefficient \( t \) through

\[
T_{\text{TE}} = \left| t_{\text{TE}} \right|^2 \frac{n_1 \cos \theta_2}{n_2 \cos \theta_1} \\
T_{\text{TM}} = \left| t_{\text{TM}} \right|^2 \frac{n_1 \cos \theta_2}{n_2 \cos \theta_1} \\
\frac{T_{\text{TE}}}{T_{\text{TM}}} = \left| \frac{t_{\text{TE}}}{t_{\text{TM}}} \right|^2
\]
Amplitude Vs. Power Terms

Wave Amplitudes
The reflection and transmission coefficients, $r$ and $t$, relate the amplitudes of the reflected and transmitted waves relative to the applied wave. They are complex numbers because both the magnitude and phase of the wave can change at an interface.

$$E_{\text{ref}} = rE_{\text{inc}} \quad E_{\text{tm}} = tE_{\text{inc}}$$

Wave Power
The reflectance and transmittance, $R$ and $T$, relate the power of the reflected and transmitted waves relative to the applied wave. They are real numbers bound between zero and one.

$$|E_{\text{ref}}|^2 = R \cdot |E_{\text{inc}}|^2 \quad |E_{\text{tm}}|^2 = T \cdot |E_{\text{inc}}|^2$$

Often, these quantities are expressed on the decibel scale

$$R_{\text{dB}} = 10 \log_{10} (R) \quad T_{\text{dB}} = 10 \log_{10} (T)$$

Conservation of Power
When electromagnetic wave is applied to a device, it can be absorbed (i.e. converted to another form of energy), reflected and/or transmitted. Without a nuclear reaction, nothing else can happen.

$$A + R + T = 1$$

Reflectance, $R$
Fraction of power from the applied wave that is reflected from the device.

Transmittance, $T$
Fraction of power from the applied wave that is transmitted through the device.

Absorptance, $A$
Fraction of power from the applied wave that is absorbed by the device.
The Critical Angle (Total Reflection)

Above the critical angle $\theta_c$, reflection is 100%

\[
|r_{TE}| = \left| \frac{\eta_2 \cos \theta - \eta_1 \cos \theta_c}{\eta_2 \cos \theta_c + \eta_1 \cos \theta} \right| = 1
\]

\[
|r_{TM}| = \left| \frac{\eta_2 \cos \theta - \eta_1 \cos \theta}{\eta_1 \cos \theta + \eta_2 \cos \theta} \right| = 1
\]

This will happen when $\cos(\theta_c)$ is imaginary. The condition for the critical angle is derived from Snell’s Law.

\[
\theta_\parallel \geq \theta_c = \sin^{-1} \left( \frac{n_2}{n_1} \right)
\]

Condition for Total Internal Reflection (TIR)

Brewster’s Angle (Total Transmission)

**TE Polarization**

\[
r_{TE} = \frac{\eta_2 \cos \theta - \eta_1 \cos \theta}{\eta_2 \cos \theta_\parallel + \eta_1 \cos \theta} = 0 \Rightarrow \sin \theta_\parallel = \sqrt{\frac{\varepsilon_2 - \mu_2}{\varepsilon_1 \mu_2}} \leq \frac{\varepsilon_2}{\varepsilon_1} \leq \frac{\mu_1}{\mu_2}
\]

We see that as long as $\mu_1 \neq \mu_2$ then there is no Brewster’s angle.

Generally, most materials have a very weak magnetic response and there is no Brewster’s angle for TE polarized waves.

**TM Polarization**

\[
r_{TM} = \frac{\eta_2 \cos \theta - \eta_1 \cos \theta}{\eta_2 \cos \theta_\parallel + \eta_1 \cos \theta} = 0 \Rightarrow \sin \theta_\parallel = \sqrt{\frac{\varepsilon_2 - \mu_2}{\varepsilon_1 \mu_2}} \leq \frac{\varepsilon_2}{\varepsilon_1} \leq \frac{\mu_1}{\mu_2}
\]

We see that if $\varepsilon_1 = \varepsilon_2$ then there is no Brewster’s angle.

For materials with no magnetic response, the Brewster’s angle equation reduces to

\[
\tan \theta_\parallel = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \quad \varepsilon_1 \geq \varepsilon_2 \quad \text{This is the most well known equation.}
\]
Notes on a Single Interface

• It is a change in impedance that causes reflections
• Law of reflection says the angle of reflection is equal to the angle of incidence.
• Snell’s Law quantifies the angle of transmission as a function of angle of incidence and the material properties.
• Angle of transmission and reflection do not depend on polarization.
• The Fresnel equations quantify the amount of reflection and transmission, but not the angles.
• Amount of reflection and transmission depends on the polarization and angle of incidence.
• For incident angles greater than the critical angle, a wave will be completely reflected regardless of its polarization.
• When a wave is incident at the Brewster’s angle, a particular polarization will be completely transmitted.

Visualization of Wave Scattering at an Interface
Longitudinal Component of the Wave Vector

1. Boundary conditions require that the tangential component of the wave vector is continuous across the interface.

Assuming $k_x$ is purely real in material 1, $k_x$ will be purely real in material 2.

→ We have oscillations and energy flow in the $x$ direction.

2. Knowing that the dispersion relation must be satisfied, the longitudinal component of the wave vector in material 2 is calculated from the dispersion relation in material 2.

$$k_{x,2}^2 + k_{y,2}^2 = (k_0 n_2)^2$$

$\downarrow$

$$k_{y,2} = \sqrt{(k_0 n_2)^2 - k_{x,2}^2}$$

We see that $k_y$ will be purely real if $k_0 n_2 > |k_{x,2}|$.

We see that $k_y$ will be purely imaginary if $k_0 n_2 < |k_{x,2}|$.

Field at an Interface Above and Below the Critical Angle (Ignoring Reflections)

1. The field always penetrates material 2, but it may not propagate.

2. Above the critical angle, penetration is greatest near the critical angle.

3. Very high spatial frequencies are supported in material 2 despite the dispersion relation.

4. In material 2, energy always flows along $x$, but not necessarily along $y$. 

1. $n_1 < n_2$

   No critical angle

2. $n_1 > n_2$

   $\vartheta_1 < \vartheta_c$ 

   $\vartheta_1 > \vartheta_c$
Simulation of Reflection and Transmission at a Single Interface (\(n_1<n_2\))

\[n_1=1.0, \quad n_2=1.73 \rightarrow \theta_B=60^\circ\]

Simulation of Reflection and Transmission at a Single Interface (\(n_1>n_2\))

\[n_1=1.41, \quad n_2=1.0 \rightarrow \theta_C=45^\circ\]
Electromagnetic Tunneling

If an evanescent field touches a medium with higher refractive index, the field may no longer be cutoff and become a propagating wave.

This is a very unusual phenomenon because the evanescent field is contributing to power flow.

This is called electromagnetic tunneling and is analogous to electron tunneling through thin insulators.
Image Theory

When fields are symmetric in some manner about a plane, it is only necessary to calculate one half of the field because the other half contains only redundant information. Sometimes more than one plane of symmetry can be identified. Image theory can dramatically reduce the numerical size of the model being solved.

Summary of Image Theory

Perfect Electric Conductor (PEC)
Perfect Magnetic Conductor (PMC)

Electric Fields
Magnetic Fields

Image Fields

Duality

Image Theory Applied to an Airplane