Lecture Outline

- Transformation Optics
  - Coordinate transformations
  - Form invariance of Maxwell’s equations
  - Transformation electromagnetics
  - Stretching space
  - Cloaking
  - Carpet cloaking
  - Other applications
  - Concluding remarks
- Conformal Mapping
Transformation Optics

Design Process Using Spatial Transforms

Step 1 of 4:
Define Spatial Transform
Design Process Using Spatial Transforms

Step 2 of 4:
Calculate Effective Material Properties

\[ \nabla \times \vec{E} = -j\omega \mu \vec{H} \]
\[ \nabla \times \vec{H} = j\omega \varepsilon \vec{E} \]

Design Process Using Spatial Transforms

Step 3 of 4:
Map Properties to Engineered Materials
Design Process Using Spatial Transforms

Step 4 of 4: Generate Overall Lattice

Coordinate Transformation
Coordinate Transformation

We can map one coordinate space into another

\[ \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \quad \quad \vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}' \]

Jacobian Matrix \([J]\)

To aid in the coordinate transform, we use the Jacobian transformation matrix.

\[ [J] = (\nabla \vec{r}') = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix} \]

Each term quantifies the “stretching” of the coordinates.

The Jacobian matrix does not perform a coordinate transformation. It transforms functions and operations between different coordinate systems.

Gradient of a vector is a tensor! Wow! 😊
General Form of the Jacobian Matrix $[J]$

$$[J] = \begin{bmatrix}
\frac{h_1'}{h_1} \frac{\partial x_1'}{\partial x_1} & \frac{h_1'}{h_1} \frac{\partial x_1'}{\partial x_2} & \frac{h_1'}{h_1} \frac{\partial x_1'}{\partial x_3} \\
\frac{h_2'}{h_2} \frac{\partial x_2'}{\partial x_1} & \frac{h_2'}{h_2} \frac{\partial x_2'}{\partial x_2} & \frac{h_2'}{h_2} \frac{\partial x_2'}{\partial x_3} \\
\frac{h_3'}{h_3} \frac{\partial x_3'}{\partial x_1} & \frac{h_3'}{h_3} \frac{\partial x_3'}{\partial x_2} & \frac{h_3'}{h_3} \frac{\partial x_3'}{\partial x_3}
\end{bmatrix}$$

$$h_i = \sqrt{\sum_{k=1}^{n} \left( \frac{\partial x_k}{\partial x_i} \right)^2}$$

### Coordinate System

<table>
<thead>
<tr>
<th>Coordinate System</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$h_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian $(x,y,z)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Cylindrical $(\rho,\phi,z)$</td>
<td>1</td>
<td>$\rho$</td>
<td>1</td>
</tr>
<tr>
<td>Spherical $(r,\phi,\theta)$</td>
<td>1</td>
<td>$r$</td>
<td>$rsin\theta$</td>
</tr>
</tbody>
</table>

Example #1: Cylindrical to Cartesian

The Cartesian and cylindrical coordinates are related through

$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

The Jacobian matrix is then

$$[J] = \begin{bmatrix}
\cos \phi & -\rho \sin \phi & 0 \\
\sin \phi & \rho \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}$$

$$\frac{\partial x}{\partial \rho} = \cos \phi \quad \frac{\partial x}{\partial \phi} = -\rho \sin \phi \quad \frac{\partial x}{\partial z} = 0$$
$$\frac{\partial y}{\partial \rho} = \sin \phi \quad \frac{\partial y}{\partial \phi} = \rho \cos \phi \quad \frac{\partial y}{\partial z} = 0$$
$$\frac{\partial z}{\partial \rho} = 0 \quad \frac{\partial z}{\partial \phi} = 0 \quad \frac{\partial z}{\partial z} = 1$$
Example #2: Spherical to Cartesian

The Cartesian and spherical coordinates are related through

\[
\begin{align*}
  x &= r \sin \theta \cos \phi \\
  y &= r \sin \theta \sin \phi \\
  z &= r \cos \theta \\
\end{align*}
\]

The Jacobian matrix is then

\[
\begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{bmatrix}
\]

Example #3: Cartesian to Cartesian

Coordinates in Cartesian space can be transformed according to

\[
\begin{align*}
  x' &= x'(x, y, z) \\
  y' &= y'(x, y, z) \\
  z' &= z'(x, y, z)
\end{align*}
\]

The Jacobian matrix is defined as

\[
[J] = \begin{bmatrix}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\
\frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\
\frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z}
\end{bmatrix}
\]
Transforming Vector Functions

A vector function (variable that changes as a function of position) in two coordinate systems is related through the Jacobian matrix as follows.

\[
\vec{E}'(\vec{r}') = \left( [J]^T \right)^{-1} \vec{E}(\vec{r})
\]

\[
\vec{E}(\vec{r}) = [J]^T \vec{E}'(\vec{r}')
\]

Transforming Operations

An operation (think derivatives, integrals, tensors, etc.) can also be transformed between two coordinate systems using the Jacobian matrix.

\[
\begin{bmatrix} F'(\vec{r}') \end{bmatrix} = \frac{[J][F(\vec{r})][J]^T}{\det[J]}
\]

\[
\begin{bmatrix} F(\vec{r}) \end{bmatrix} = \det[J] \cdot [J]^{-1} \left[ F'(\vec{r}') \right] \left( [J]^T \right)^{-1}
\]
Form Invariance of Maxwell’s Equations

Maxwell’s Equations are Form Invariant

In ANY coordinate system, Maxwell’s equations can be written as

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>Cartesian</td>
<td>$\nabla \times \vec{H} = j\omega \varepsilon \vec{E}$</td>
<td>$\nabla \times \vec{E} = -j\omega \mu \vec{H}$</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>$\nabla \times \vec{H} = j\omega \varepsilon \vec{E}$</td>
<td>$\nabla \times \vec{E} = -j\omega \mu \vec{H}$</td>
</tr>
<tr>
<td>Spherical</td>
<td>$\nabla \times \vec{H} = j\omega \varepsilon \vec{E}$</td>
<td>$\nabla \times \vec{E} = -j\omega \mu \vec{H}$</td>
</tr>
<tr>
<td>Martian</td>
<td>$\nabla \times \vec{H} = j\omega \varepsilon \vec{E}$</td>
<td>$\nabla \times \vec{E} = -j\omega \mu \vec{H}$</td>
</tr>
</tbody>
</table>

We can transform Maxwell’s equations to a different coordinate system, but they still have the same form.

\[
\nabla' \times \vec{H}' = j\omega \varepsilon' \vec{E}'
\n\nabla' \times \vec{E}' = -j\omega \mu' \vec{H}'
\]
Important Consequence

We can “absorb” the coordinate transformation completely into the material properties.

\[ \nabla' \times \vec{H}' = j \omega [ \varepsilon' ] \vec{E}' \]

\[ \nabla' \times \vec{E}' = -j \omega [ \mu' ] \vec{H}' \]

\[ \nabla \times \vec{H} = j \omega [ \varepsilon^* ] \vec{E} \]

\[ \nabla \times \vec{E} = -j \omega [ \mu^* ] \vec{H} \]

We are now back to the original coordinates, but the fields behave almost as if they are in the transformed coordinates.

Absorbing the Coordinate Transformation into the Materials

Given the Jacobian \([J]\) describing the coordinate transformation, the material property tensors are related through

\[ [\mu'] = \frac{[J][\mu][J]^T}{\det[J]} \]

\[ [\varepsilon'] = \frac{[J][\varepsilon][J]^T}{\det[J]} \]

Here we are actually transforming an operation, not a function.

\[ [J] = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix} \]

We can think of this as a “stretching” matrix. It describes how much the coordinate changes in our transformed system with respect to a change in the original system.
Proof of Form Invariance (1 of 3)

We need to show that the following transform is true.

$$\nabla \times \vec{E} = -j\omega [\mu] \vec{H} \quad \rightarrow \quad \nabla' \times \vec{E}' = -j\omega [\mu'] \vec{H}'$$

Defining the coordinate transformation as $$\vec{r}' = \vec{r}'(\vec{r})$$, we have

$$\vec{E}'(\vec{r}') = \left([J]^{T}\right)^{-1} \vec{E}(\vec{r})$$

$$\vec{H}'(\vec{r}') = \left([J]^{T}\right)^{-1} \vec{H}(\vec{r})$$

$$[\mu'(\vec{r}')] = \left[\frac{[J][\mu(\vec{r})][J]^{T}}{\det[J]} \right]$$

Proof of Form Invariance (2 of 3)

We substitute our transforms into the curl equation.

$$\nabla \times \vec{E} = -j\omega [\mu] \vec{H}$$

$$\vec{E}(\vec{r}) = [J]^{T} \vec{E}'(\vec{r}')$$

$$\vec{H}(\vec{r}) = [J]^{T} \vec{H}'(\vec{r}')$$

$$[\mu(\vec{r})] = \det[J] \left[\frac{[J][\mu'(\vec{r})][J]^{T}}{\det[J]} \right]$$

This becomes

$$\nabla \times \left([J]^{T} \vec{E}'\right) = -j\omega \det[J] \left[\frac{[J][\mu'][[J]^{T}]^{-1}}{\det[J]} \right] \left[J\right]^{T} \vec{H}'$$

$$\downarrow$$

$$\left[\frac{[J][\nabla \times][J]^{T}}{\det[J]}\right] \vec{E}' = -j\omega [\mu'] \vec{H}'$$
Proof of Form Invariance (3 of 3)

Recall the form of transforming an operation

\[
[F'(\mathbf{r}')] = \frac{[J][F(\mathbf{r})][J]^T}{\det[J]}
\]

We see that the group of terms around the curl operation indicates this is just the transformed curl.

\[
\frac{[J][\nabla \times][J]^T}{\det[J]} = \frac{\nabla' \times \tilde{E}' = -j\omega[\mu']\tilde{H}'}{\nabla' \times \tilde{E}' = -j\omega[\mu']\tilde{H}'}
\]

A Simple Example of the Proof

Start with Maxwell’s curl equation.

\[
\nabla \times \tilde{E}(\mathbf{r}) = -j\omega[\mu(\mathbf{r})]\tilde{H}(\mathbf{r})
\]

We define the following coordinate transform

\[
\mathbf{r}' = a\mathbf{r}
\]

The terms transform according to

\[
\tilde{E}(\mathbf{r}) \rightarrow \tilde{E}'(\mathbf{r}')
\]

\[
[\mu(\mathbf{r})] \rightarrow [\mu'(\mathbf{r}')] \quad \nabla \times = \begin{bmatrix}
0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} \\
-\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0
\end{bmatrix} \rightarrow \nabla' \times = \begin{bmatrix}
0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} \\
-\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0
\end{bmatrix}
\]

The scale factor from the curl operation can be absorbed into the permeability.

\[
\nabla \times \tilde{E}(\mathbf{r}) = -j\omega[a\mu(\mathbf{r})]\tilde{H}(\mathbf{r})
\]
Analytical Transformation Electromagnetics

Concept of Transformation EM

Transformation electromagnetics is an analytical technique to calculate the permittivity and permeability functions that will bend fields in a prescribed manner.

1. Define a uniform grid with uniform rays.
2. Perform a coordinate transformation such that the rays follow some desired path.
3. Move the coordinate transformation into the material tensors.

Lecture 16a – Spatial Transforms
Step 1: Pick a Coordinate System

Pick a coordinate system that is most convenient for your device geometry.

Step 2: Draw Straight Rays Through the Grid

Assuming we start with a wave in free space, draw the rays passing straight through the coordinate system.

\[ \begin{bmatrix} \mu \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
Step 3: Define a Coordinate Transform

Define a coordinate transform so that the rays will follow the desired path. Here, we are “squeezing” the wave at the center of the grid.

\[ x' = x \]
\[ y' = y \left[ 1 - \exp \left( -\frac{x^2}{\sigma_x^2} - \frac{y^2}{\sigma_y^2} \right) \right] \]
\[ z' = z \]

Step 4: Calculate the Jacobian Matrix

Given the coordinate transformation, the Jacobian matrix is calculated.

\[ J = \begin{bmatrix}
1 & 0 & 0 \\
\frac{2xy}{\sigma_x^2}e^{-\frac{x^2}{\sigma_x^2}} - \frac{2y^2}{\sigma_y^2} & 1 + \left( \frac{2y^2}{\sigma_y^2} - 1 \right) & 0 \\
0 & 0 & 1
\end{bmatrix} \]
Step 5: Calculate the Material Tensors

The material tensors are calculated according to.

\[
\mathbf{\mu}' = \frac{[J][\mu][J]^T}{\det[J]} \quad \mathbf{\epsilon}' = \frac{[J][\epsilon][J]^T}{\det[J]}
\]

This is

\[
\mu_{xx} = \epsilon_{xx} = \mu_{yy} = \epsilon_{yy} = \frac{\sigma_x^2}{\sigma_y^2 + (2y^2 - \sigma_z^2)\epsilon_x \epsilon_y \epsilon_z}
\]

\[
\mu_{xy} = \mu_{yx} = \epsilon_{xy} = \epsilon_{yx} = \frac{2xy\sigma_x^2}{\sigma_y^2 - \sigma_z^2 + \epsilon_x \epsilon_y \epsilon_z}
\]

\[
\mu_{xz} = \epsilon_{xz} = \epsilon_{zx} = \frac{\sigma_x^2 \epsilon_x \epsilon_y \epsilon_z}{\left(\frac{2y^2}{\sigma_x^2} - \frac{x^2}{\sigma_x^2} - \frac{z^2}{\sigma_z^2} + \frac{1}{\sigma_y^2}\right)^2 + \frac{4y^2z^2}{\sigma_x^2 \sigma_y^2} \epsilon_x \epsilon_y \epsilon_z}
\]

\[
\mu_{yz} = \epsilon_{yz} = \epsilon_{zy} = \mu_{zz} = \epsilon_{zz} = \mu_{zy} = \epsilon_{zy} = 0
\]

Plot of the Final Tensor Over Entire Device
MATLAB Code to Generate the Equations (Analytical Approach)

```matlab
% DEFINE VARIABLES
syms x y z;
syms sx sy;

% INITIALIZE MATERIALS
UR = eye(3,3);
ER = eye(3,3);

% DEFINE COORDINATE TRANSFORMATION
xp = x;
yp = y*(1 - exp(-x^2/sx^2)*exp(-y^2/sy^2));
zp = z;

% COMPUTE ELEMENTS OF JACOBIAN
J = [diff(xp,x) diff(xp,y) diff(xp,z); ...
     diff(yp,x) diff(yp,y) diff(yp,z); ...
     diff(zp,x) diff(zp,y) diff(zp,z)];

% ABSORB TRANSFORM INTO MATERIALS
UR = J*UR*J.'/det(J);
ER = J*ER*J.'/det(J);

% SHOW TENSOR
pretty(ER);
```

MATLAB Code to Fill Grid
(Analytical Approach)

```matlab
% PARAMETERS
ER = subs(ER,sx,0.25);
ER = subs(ER,sy,0.25);

% FILL GRID
for ny = 1 : Ny
    for nx = 1 : Nx
        er = subs(ER,x,X(nx,ny));
        er = subs(er,y,Y(nx,ny));
        ERxx(nx,ny) = er(1,1);
        ERxy(nx,ny) = er(1,2);
        ERxz(nx,ny) = er(1,3);
        ERyx(nx,ny) = er(2,1);
        ERyy(nx,ny) = er(2,2);
        ERyz(nx,ny) = er(2,3);
        ERzx(nx,ny) = er(3,1);
        ERzy(nx,ny) = er(3,2);
        ERzz(nx,ny) = er(3,3);
    end
end
```

This is VERY slow!!!

Recommend calculating analytical equations and then manually typing these into MATLAB to construct the device.
Stretching Space

Setup of the Problem

Starting coordinate system

We want to "stretch" space in order to move two objects farther apart.

We calculate metamaterial properties that will do this.
Define the Coordinate Transform

Suppose we want to “stretch” the z-axis by a factor of $a$.

$x' = x$  
$y' = y$  
$z' = z/a$

We wish to end in the standard uniform Cartesian coordinate system. This means we must start in a coordinate system that is stretched. Therefore, the coordinate transform must compress space. This will give $[\mu]$ and $[\epsilon]$ that stretch space.

Calculate the Jacobian

$$x' = x$$  
$$y' = y$$  
$$z' = z/a$$

$$[J] = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{bmatrix}$$
Calculate $\mu$ and $\varepsilon$

$$[\mu'] = \frac{[J][\mu][J]^t}{\det[J]} = \begin{bmatrix} 1 & 0 & 0 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 & \mu & 0 \\ 0 & 0 & a & 0 & \mu & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{bmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu/\alpha^2 \end{bmatrix} = \frac{1}{\alpha^2} \begin{bmatrix} a\mu & 0 & 0 \\ 0 & a\mu & 0 \\ 0 & 0 & \mu/\alpha \end{bmatrix}$$

$$[\varepsilon'] = \frac{[J][\varepsilon][J]^t}{\det[J]} = \begin{bmatrix} 1 & 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 1 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & a & 0 & \varepsilon & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/a \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon/\alpha^2 \end{bmatrix} = \frac{1}{\alpha^2} \begin{bmatrix} a\varepsilon & 0 & 0 \\ 0 & a\varepsilon & 0 \\ 0 & 0 & \varepsilon/\alpha \end{bmatrix}$$

What Does the Answer Mean?

$$[\mu'] = \begin{bmatrix} a\mu & 0 & 0 \\ 0 & a\mu & 0 \\ 0 & 0 & \mu/\alpha \end{bmatrix} \quad [\varepsilon'] = \begin{bmatrix} a\varepsilon & 0 & 0 \\ 0 & a\varepsilon & 0 \\ 0 & 0 & \varepsilon/\alpha \end{bmatrix}$$

These tensors have two equal elements and a different third element. $\rightarrow$ Uniaxial

For $a > 0$, the third element is smaller in value than the first two. $\rightarrow$ Negative uniaxial
Recall from Lecture 13, a negative uniaxial metamaterial is an array of sheets.

\[
[e'] = \begin{bmatrix}
  \varepsilon_o & 0 & 0 \\
  0 & \varepsilon_o & 0 \\
  0 & 0 & \varepsilon_e
\end{bmatrix}
\]

\[\varepsilon_o = a\varepsilon\]
\[\varepsilon_e = \varepsilon/\alpha\]

Stretching factor given tensor:
\[a = \sqrt{\varepsilon_o/\varepsilon_e}\]
\[\varepsilon = \sqrt{\varepsilon_o\varepsilon_e}\]
What is Cloaking?

A true cloak must have the following properties:
1. Cannot reflect or scatter waves.
2. Must perfectly reconstruct the wave front on the other side of the object.
3. Must work for waves applied from any direction.


Famous Cylinder Cloak

Figure 3. A two-dimensional circular annular cloak: (a) A conducting cylinder subject to a TM-mode plane-wave illumination, (b) the same conducting cylinder enclosed within a cloak.
To render a region in the center of the grid invisible, the wave must be made to flow around it.

Figure 2. A coordinate transformation for a two-dimensional circular annular cloak: (a) the original coordinate system, (b) the transformed coordinate system.
Deriving the Coordinate Transformation

Figure 2. A coordinate transformation for a two-dimensional circular annular disk: (a) the original coordinate system, (b) the transformed coordinate system.

This transform can be done with a simple straight line.

\[ y = mx + b \quad \rightarrow \quad \rho' = m \rho + b \]

- **y-intercept:** \( b = R_1 \)
- **Slope:** \( m = \frac{R_2 - R_1}{R_2} \)

It follows that the Jacobian matrix is

\[
[J] = \begin{bmatrix}
1 & 1 \frac{\partial \rho'}{\partial \rho} & 1 \frac{\partial \rho'}{\partial \phi} \\
1 & 1 \frac{\partial \phi'}{\partial \rho} & 1 \frac{\partial \phi'}{\partial \phi} \\
1 \frac{\partial z'}{\partial \rho} & 1 \frac{\partial z'}{\partial \phi} & 1 \frac{\partial z'}{\partial \phi}
\end{bmatrix} = \begin{bmatrix}
(R_2 - R_1)/R_2 & 0 & 0 \\
0 & \rho' / \rho & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The Jacobian Matrix

Due to the cylindrical geometry, the coordinate transformation will be in cylindrical coordinates.

\[ \rho' = R_1 + \frac{R_2 - R_1}{R_2} \rho \]

\[ \phi' = \phi \]

\[ z' = z \]

It follows that the Jacobian matrix is
The Material Tensors

The material tensors are

\[
\begin{bmatrix}
\mu'(\vec{r}') \\
\varepsilon'(\vec{r}')
\end{bmatrix} = \frac{[J][\mu(\vec{r})][J]^T}{\det[J]} \quad \frac{[J][\varepsilon(\vec{r})][J]^T}{\det[J]}
\]

Assuming we start with free space, we will have

\[
\begin{bmatrix}
\mu'(\vec{r}') \\
\varepsilon'(\vec{r}')
\end{bmatrix} = \begin{bmatrix}
\frac{r' - R_1}{r'} & 0 & 0 \\
0 & \frac{r'}{r' - R_1} & 0 \\
0 & 0 & \left(\frac{R_2}{R_2 - R_1}\right)^2 & \frac{r' - R_1}{r'}
\end{bmatrix}
\]

Cylindrical Tensors

\[
\begin{align*}
\mu'_r (\vec{r}') &= \varepsilon'_r (\vec{r}') \\
\mu'_{\theta\theta} (\vec{r}') &= \varepsilon'_{\theta\theta} (\vec{r}') \\
\mu'_{zz} (\vec{r}') &= \varepsilon'_{zz} (\vec{r}')
\end{align*}
\]

\[
\begin{align*}
\frac{r' - R_1}{r'} & \quad \frac{r'}{r' - R_1} & \quad \left(\frac{R_2}{R_2 - R_1}\right)^2 & \quad \frac{r' - R_1}{r'}
\end{align*}
\]

\[ R_1 = 0.10 \quad R_2 = 0.45 \]
Carpet Cloaking

Problems with Cloaking by Transformation Electromagnetics

• The resulting materials:
  – Are anisotropic
  – Require dielectric and magnetic properties
  – Require extreme and singular values
• Particularly problematic at optical frequencies
Three Cases for Cloaking

Squish an object to a point.

Squish an object to a line.

Object becomes infinitely conducting.
This is not a problem because the objects have zero size.
These can be rendered invisible, but require extreme and singular values as well as being anisotropic.

Squish an object to a sheet.

A sheet is highly visible.
It can only be made invisible if it sits on another conducting sheet so they cannot be distinguished.
While more limited, invisibility can be realized without extreme values and with isotropic materials.

Carpet Cloak Concept

FIG. 1 (color online). The virtual and the physical systems. The regions in cyan are transformed into each other. Shaded regions represent the ground planes. The observer perceives the physical system as the virtual one with a flat ground plane.

The Jacobian Matrix and Covariant Matrix

We define the Jacobian matrix $[J]$ just like we did before.

$$A_y = \frac{\partial x_i}{\partial x'_i}$$

The Covariant matrix is then

$$[g] = [J]^{T} [J]$$

The Original System
The Transformed System

The Transformation

\[ \varepsilon = \frac{\varepsilon_{\text{ref}}}{\sqrt{\det[g]}} \]

\[ [\mu] = \frac{[J][J]^T}{\sqrt{\det[g]}} \]

\[ \varepsilon_{\text{ref}} \text{ anything} \]

\[ \mu_{\text{ref}} = 1 \]
The permeability is still a tensor quantity. For a given wave, it will have two principle values.

\[ [\mu] \rightarrow \mu_T \text{ and } \mu_L \quad \mu_T \mu_L = 1 \]

This gives us two refractive indices to characterize the medium.

\[ n_T = \sqrt{\mu_T \varepsilon} \quad n_L = \sqrt{\mu_L \varepsilon} \]

We can characterize the anisotropy using the anisotropy factor \( \alpha \).

\[ \alpha = \max \left[ \frac{n_T}{n_L}, \frac{n_L}{n_T} \right] \]

By picking a suitable transform, we can neglect the permeability.

\[ \varepsilon = \frac{\varepsilon_{ref}}{\sqrt{\det[g]}} \quad [\mu] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

The optimal transformation is generated by minimizing the Modified-Liao functional

\[ \Phi = \frac{1}{hw} \int_0^w \int_0^h \frac{(\text{Tr}[g])^2}{\det[g]} \, dy' \, dx' \]

This leads to high \( \varepsilon \) and low anisotropy.
Impact of Grid on Isotropic Approximation

Low $\varepsilon$ so this medium will be poorly approximated as isotropic.

High $\varepsilon$ so this medium is well approximated as isotropic.

Example Carpet Cloak

Scattering from cloaked object.

Scattering from just the object.
An antenna can be cloaked to either render it invisible to a “bad guy” or to reduce the effects of scattering and coupling to nearby objects.

This implies a “dual band” mode of operation because the cloak must be transparent to the radiation frequency of the antenna.

Metamaterials used to realize cloaks are inherently narrowband which is an advantage for this application.
Electromagnetic Cloaking of Antennas (2 of 2)

Antenna $A_1$ transmits at frequency $f_1$.
Antenna $A_2$ transmits at frequency $f_2$.
Cloak of $A_1$ must be transparent to $f_1$.
Cloak of $A_2$ must be transparent to $f_2$.

Scattering of an elliptical shaped object.
Rectangular object embedded in an anisotropic medium designed by TEM to scatter like the elliptical object.

Polarization Splitter

Device is made uniform in the \( z \) direction.

Maxwell's equations split into two independent modes.

Each mode can be independently controlled.

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Polarization Rotator

Figure 13: The coordinate transformation employed to design a beam-polarization rotator: (a) the original coordinate system, \((x, y, z)\); (b) the transformed coordinate system. The entire circular cylindrical column is transformed, not only the cylindrical shell in displayed for clarity of illustration.

Figure 14: Snapshots of the total electric-field distribution in the \( xy \) and \( xz \) planes (702 ± 60 \( E_x \), \( 200 \) \( E_z \)). An \( E \)-polarized Gaussian beams impinged upon the device from the \(-z\) direction.
Wave Collimator

Figure 16. Snapshots of the $\pi$-polarized total electric-field distribution due to an electric line source located at the coordinate origin: (a) with the line source radiating in free space, (b) with the line source embedded in the wave collimator.

Flat Lenses

Figure 17. The coordinate transformation for a two-dimensional far-zone-focusing lens design: (a) the original coordinate system, (b) the transformed coordinate system.

Figure 18. Snapshots of the total electric field for a line source radiating at the coordinate origin: (a) for the line source radiating in free space, (b) for the line source radiating in the presence of the far-zone-focusing flat lens.
Concluding

Remarks
Notes

- Transformation electromagnetics (TEM) is a coordinate transformation technique where the transformation is “absorbed” into the permittivity and permeability functions.
- The method produces the permittivity and permeability as a function of position.
- The method does not say anything about how the materials will be realized.
- The resulting material functions will be anisotropic and require dielectric and magnetic metamaterials with often extreme values. This leads to structures requiring metals.
- TEM is perhaps more commonly called transformation optics (TO) because that is where it first appeared.

Conformal Mapping
Observations About TO

**Observation #1**
Sharper bends lead to more extreme material properties.

**Observation #2**
Oblique coordinates lead to anisotropic material properties.

Complex Numbers as Coordinates

\[
\text{Re}[u] \quad \text{Im}[u]
\]
Coordinate Transform with Complex Numbers

Observe the constant orthogonality of the coordinate axes.

→ Conformal mapping