Lecture #9

Diffraction Gratings

Lecture Outline

- Fourier series
- Diffraction from gratings
- The plane wave spectrum
- Plane wave spectrum for crossed gratings
- The grating spectrometer
- Littrow gratings
- Patterned fanout gratings
- Diffractive optical elements
Fourier Series

Jean Baptiste Joseph Fourier

Born: March 21, 1768
in Yonne, France.
Died: May 16, 1830
in Paris, France.

1D Complex Fourier Series

If a function \( f(x) \) is periodic with period \( \Lambda \), it can be expanded into a complex Fourier series.

\[
f(x) = \sum_{m=-\infty}^{\infty} a(m) e^{\frac{2\pi imx}{\Lambda}}
\]

\[
a(m) = \frac{1}{\Lambda} \int_{-\Lambda/2}^{\Lambda/2} f(x) e^{-\frac{2\pi imx}{\Lambda}} \, dx
\]

Typically, we retain only a finite number of terms in the expansion.

\[
f(x) = \sum_{m=-M}^{M} a(m) e^{\frac{2\pi imx}{\Lambda}}
\]
For 2D periodic functions, the complex Fourier series generalizes to

\[ f(x, y) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} a(p, q) e^{\frac{2\pi ip}{\lambda_x} \frac{2\pi jq}{\lambda_y}} \]

\[ a(p, q) = \frac{1}{A} \iint_{A} f(x, y) e^{-\frac{2\pi ip}{\lambda_x} \frac{2\pi jq}{\lambda_y}} dA \]

Diffraction from Gratings
Fields in Periodic Structures

Waves in periodic structures take on the same periodicity as their host.

\[ \vec{k} \]

Diffraction Orders

The field must be continuous so only discrete directions are allowed. The allowed directions are called the diffraction orders. The allowed angles are calculated using the famous grating equation.
Field in a Periodic Structure

The dielectric function of a sinusoidal grating can be written as

\[ \varepsilon_r(\vec{r}) = \varepsilon_{r,\text{avg}} + \Delta \varepsilon \cos(\vec{K} \cdot \vec{r}) \]

A wave propagating through this grating takes on the same symmetry.

\[ E(\vec{r}) = A(\vec{r}) e^{-j\beta_{m,\text{avg}} \cdot \vec{r}} \]

\[ = A \left[ \varepsilon_{r,\text{avg}} + \Delta \varepsilon \cos(\vec{K} \cdot \vec{r}) \right] e^{-j\beta_{m,\text{avg}} \cdot \vec{r}} \]

\[ = \ldots \]

\[ = A\varepsilon_{r,\text{avg}} e^{-j\beta_{m,\text{avg}} \cdot \vec{r}} + \frac{A\Delta \varepsilon}{2} e^{-j(k_m - K) \cdot \vec{r}} + \frac{A\Delta \varepsilon}{2} e^{-j(k_m + K) \cdot \vec{r}} \]

Grating Produces New Waves

The applied wave splits into three waves.

\[ e^{-j\beta_{m,\text{avg}} \cdot \vec{r}} \]

\[ e^{-j\beta_{m,\text{avg}} \cdot \vec{r}} \rightarrow e^{-j(k_m - K) \cdot \vec{r}} \]

\[ e^{-j\beta_{m,\text{avg}} \cdot \vec{r}} \]

Each of those splits into three waves as well.

\[ e^{-j\beta_{m,\text{avg}} \cdot \vec{r}} \rightarrow e^{-j(k_m - K) \cdot \vec{r}} \rightarrow e^{-j(k_m - 2K) \cdot \vec{r}} \rightarrow e^{-j(k_m + K) \cdot \vec{r}} \rightarrow e^{-j\beta_{m,\text{avg}} \cdot \vec{r}} \]

And each of these split, and so on.

\[ \vec{k}(m) = \vec{k}_{mc} - m\vec{K} \quad m = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty \]

This equation describes the total set of allowed harmonics.
Wave Incident on a Grating

Boundary conditions required the tangential component of the wave vector be continuous.

\[ k_{x,\text{trn}} = k_{x,\text{inc}} \]

The wave is entering a grating, so the phase matching condition is

\[ k_x(m) = k_{x,\text{inc}} - mK_x \]

The longitudinal vector component is calculated from the dispersion relation.

\[ k_z^2(m) = \left(k_0 n_{\text{avg}}\right)^2 - k_x^2(m) \]

For large \( m \), \( k_{z,m} \) can actually become imaginary. This indicates that the highest order spatial harmonics are evanescent.

The Grating Equation

\[ n_{\text{avg}} \sin[\theta(m)] = n_{\text{inc}} \sin[\theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda}] \sin \phi \]

Note, this really is just

\[ k_x(m) = k_{x,\text{inc}} - mK_x \]

Proof:

\[ k_0 n_{\text{avg}} \sin[\theta(m)] = k_0 n_{\text{inc}} \sin[\theta_{\text{inc}} - m \frac{2\pi}{\Lambda}] \]

\[ \frac{2\pi}{\Lambda_0} n_{\text{avg}} \sin[\theta(m)] = \frac{2\pi}{\Lambda_0} n_{\text{inc}} \sin[\theta_{\text{inc}} - m \frac{2\pi}{\Lambda}] \]

\[ n_{\text{avg}} \sin[\theta(m)] = n_{\text{inc}} \sin[\theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda}] \]

\[ n_{\text{avg}} \sin[\theta(m)] = n_{\text{inc}} \sin[\theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda}] \sin \phi \]
Grating Equation in Different Regions

The angles of the diffracted modes are related to the wavelength and grating through the grating equation.

The grating equation only predicts the directions of the modes, not how much power is in them.

**Reflection Region**
\[ n_{\text{ref}} \sin[\theta(m)] = n_{\text{inc}} \sin \theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda_x} \]

**Transmission Region**
\[ n_{\text{trn}} \sin[\theta(m)] = n_{\text{inc}} \sin \theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda_x} \]

Diffraction in Two Dimensions

- We know everything about the direction of diffracted waves just from the grating period.

- The grating equation says nothing about how much power is in the diffracted modes.
  - We need to solve Maxwell’s equations for that!
Effect of Grating Periodicity

Subwavelength Grating

\( \Lambda_s < \frac{\lambda_0}{n_{\text{avg}}} \)

“Subwavelength” Grating

\( \frac{\lambda_0}{n_{\text{avg}}} < \Lambda_s < \frac{\lambda_0}{n_{\text{inc}}} \)

Low Order Grating

\( \Lambda_s > \frac{\lambda_0}{n_{\text{inc}}} \)

High Order Grating

\( \Lambda_s >> \frac{\lambda_0}{n_{\text{inc}}} \)

Animation of Grating Diffraction at Normal Incidence

\( a/\lambda = 0.10 \)

\( a/\lambda = 0.25 \)
Wood’s Anomalies

Robert W. Wood observed rapid variations in the spectrum of light diffracted by gratings which he could not explain.

**Type 1 – Rayleigh Singularities**
Rapid variation in the amplitudes of the diffracted modes the correspond to the onset or disappearance of other diffracted modes.

R. W. Wood, Phil. Mag. 4, 396 (1902)

**Type 2 – Resonance Effects**
A resonance condition arising from leaky waves supported by the grating. Today, we call this guided-mode resonance.

Robert Williams Wood
1868 - 1955

Grating Cutoff Wavelength

When \( \theta_m \) becomes imaginary, the mode is evanescent and cut off.

Assuming normal incidence (i.e. \( \theta_{\text{inc}} = 90^\circ \)), the grating equation reduces to

\[
n \sin[\theta(m)] = -m \frac{\lambda_0}{\Lambda_x}
\]

The first diffracted modes to appear are \( m = \pm 1 \).

The cutoff for the first-order modes happens when \( \theta(\pm 1) = 90^\circ \).

\[
\theta(\pm 1) = 90^\circ \quad \text{To prevent the first-order modes, we need}
\]

\[
\sin[90^\circ] = 1 = \frac{\lambda_0}{n \Lambda_x} \quad \Lambda_x < \frac{\lambda_0}{n}
\]

\[
\Lambda_x = \frac{\lambda_0}{n} \quad \text{To ensure we have first-order modes, we need}
\]

\[
\Lambda_x > \frac{\lambda_0}{n}
\]

Total Number of Diffracted Modes

Given the grating period \( \Lambda_x \) and the wavelength \( \lambda_0 \), we can determine how many diffracted modes exist.

Again, assuming normal incidence,

\[
\sin[\theta(m)] = -m \frac{\lambda_0}{n_{\text{avg}} \Lambda_x} \quad \rightarrow \quad \sin[\theta(m)] = \left| -m \frac{\lambda_0}{n_{\text{avg}} \Lambda_x} \right| < 1
\]

Therefore, a maximum value for \( m \) is

\[
m_{\text{max}} = \frac{n_{\text{avg}} \Lambda_x}{\lambda_0}
\]

The total number of possible diffracted modes \( M \) is then \( 2m_{\text{max}} + 1 \)

\[
M = \frac{2n_{\text{avg}} \Lambda_x}{\lambda_0} + 1
\]
### Determining Grating Cutoff Conditions

<table>
<thead>
<tr>
<th>Condition</th>
<th>Requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-order mode</td>
<td>Always exists</td>
</tr>
<tr>
<td>No 1&lt;sup&gt;st&lt;/sup&gt;-order modes</td>
<td>Grating period must be shorter than what causes $\theta(\pm1) = 90^\circ$</td>
</tr>
<tr>
<td>Ensure 1&lt;sup&gt;st&lt;/sup&gt;-order modes</td>
<td>Grating period must be larger than what causes $\theta(\pm1) = 90^\circ$</td>
</tr>
<tr>
<td>No 2&lt;sup&gt;nd&lt;/sup&gt;-order modes</td>
<td>Grating period must be shorter than what causes $\theta(\pm2) = 90^\circ$</td>
</tr>
<tr>
<td>Ensure 2&lt;sup&gt;nd&lt;/sup&gt;-order modes</td>
<td>Grating period must be larger than what causes $\theta(\pm2) = 90^\circ$</td>
</tr>
<tr>
<td>No m&lt;sup&gt;th&lt;/sup&gt;-order modes</td>
<td>Grating period must be shorter than what causes $\theta(\pm m) = 90^\circ$</td>
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</tr>
</tbody>
</table>

### Three Modes of Operation for 1D Gratings

**Bragg Grating**
- Couples energy between counter-propagating waves.

**Diffraction Grating**
- Couples energy between waves at different angles.

**Long Period Grating**
- Couples energy between co-propagating waves.

#### Applications
- Thin film optical filters
- Fiber optic gratings
- Wavelength division multiplexing
- Dielectric mirrors
- Photonic crystal waveguides

- Beam splitters
- Patterned fanout gratings
- Laser locking
- Spectrometry
- Sensing
- Anti-reflection
- Frequency selective surfaces
- Grating couplers

- Sensing
- Directional coupling
**Analysis of Diffraction Gratings**

**Direction of the Diffracted Modes**

\[ n \sin \theta (m) = n_\text{inc} \sin \theta_\text{inc} - m \frac{\lambda}{\Lambda} \sin \phi \]

**Diffraction Efficiency and Polarization of the Diffracted Modes**

We must obtain a rigorous solution to Maxwell’s equations to determine amplitude and polarization of the diffracted modes.

\[
\begin{align*}
\nabla \times \vec{E} &= -j\omega \mu \vec{H} \\
\nabla \times \vec{H} &= j\omega \varepsilon \vec{E} \\
\n\nabla \cdot (\varepsilon \vec{E}) &= 0 \\
\n\nabla \cdot (\mu \vec{H}) &= 0
\end{align*}
\]

**Applications of Gratings**

**Subwavelength Gratings**

Only the zero-order modes may exist.

**Littrow Gratings**

Gratings in the littrow configuration are a spectrally selective retroreflector.

**Patterned Fanout Gratings**

Gratings diffract laser light to form images.

**Applications**

- Polarizers
- Artificial birefringence
- Form birefringence
- Anti-reflection
- Effective index media

**Littrow Gratings**

Gratings in the littrow configuration are a spectrally selective retroreflector.

**Patterned Fanout Gratings**

Gratings diffract laser light to form images.

**Applications**

- Sensors
- Lasers

**Applications**

- Spectrometry

Gratings separate broadband light into its component colors.

**Holograms**

Holograms are stored as gratings.
The Plane Wave Spectrum

Periodic Functions Can Be Expanded into a Fourier Series

Waves in periodic structures obey Bloch's equation

\[ E(x, y) = A(x) e^{i \beta \cdot \vec{r}} \]

The envelope \( A(x) \) is periodic with period \( \Lambda \), so it can be expanded into a Fourier series.

\[ E(x, y) = e^{i \beta \cdot \vec{r}} \sum_{m=-\infty}^{\infty} S(m; y) e^{-j\frac{2\pi mx}{\Lambda}} \]

\[ S(m; y) = \int_{\Lambda} A(x, y) e^{-j\frac{2\pi mx}{\Lambda}} \, dx \]
Rearrange the Fourier Series

A periodic field can be expanded into a Fourier series.

\[ E(x, y) = e^{i\beta_x r} \sum_{m=-\infty}^{\infty} S(m; y) e^{\frac{2\pi mx}{\Lambda_x}} \]

Let's define the transverse wave vector component of the \( m \)th term.

\[ k_x(m) = \beta_x - \frac{2\pi m}{\Lambda_x} \]

So the field can be written as

\[ E(x, y) = \sum_{m=-\infty}^{\infty} S(m) e^{ik_x x} e^{i\beta_y y} \]

This has the form of a sum of plane waves all at different angles.

The Plane Wave Spectrum

We rearranged terms and saw that a periodic field can also be thought of as an infinite sum of plane waves at different angles. This is the “plane wave spectrum” of a field.
The wave incident on a grating can be written as

\[ E_{\text{inc}}(x,y) = E_0 e^{i(k_x^{\text{inc}}x + k_y^{\text{inc}}y)} \]

\[ k_x^{\text{inc}} = k_0 n_{\text{inc}} \sin \theta_{\text{inc}} \]

\[ k_y^{\text{inc}} = k_0 n_{\text{inc}} \cos \theta_{\text{inc}} \]

Phase matching into the grating leads to

\[ k_x^{\text{grat}}(m) = k_x^{\text{inc}} - m \frac{2\pi}{\Lambda_x} \quad m = \cdots, -2, -1, 0, 1, 2, \cdots \]

Note: \( k_y \) is always real.

Each wave must satisfy the dispersion relation.

\[
\left[k_x^{\text{grat}}(m)\right]^2 + \left[k_y^{\text{grat}}(m)\right]^2 = \left(k_0 n_{\text{grat}}\right)^2
\]

We have two possible solutions here.

1. Purely real \( k_y \)
2. Purely imaginary \( k_y \)

Visualizing Phase Matching into the Grating

The wave vector expansion for the first 11 modes can be visualized as...

\[ k_x^{\text{inc}} \]

\[ k_1^{\text{inc}} \]

\[ k_2^{\text{inc}} \]

\[ k_3^{\text{inc}} \]

\[ k_4^{\text{inc}} \]

\[ k_5^{\text{inc}} \]

\[ k_{-1}^{\text{inc}} \]

\[ k_{-2}^{\text{inc}} \]

\[ k_{-3}^{\text{inc}} \]

\[ k_{-4}^{\text{inc}} \]

\[ k_{-5}^{\text{inc}} \]

Each of these is phase matched into material 2. The longitudinal component of the wave vector is calculated using the dispersion relation in material 2.

\[ n_1 \]

\[ n_2 \]

\[ k_x \]

\[ k_y \]

Note: The “evanescent” fields in material 2 are not completely evanescent. They have a purely real \( k_x \) so they do flow energy in the transverse direction.
Conclusions

• Fields in periodic media take on the same periodicity as the media they are in.
• Periodic fields can be expanded into a Fourier series.
• Each term of the Fourier series represents a spatial harmonic (plane wave).
• Since there are in infinite number of terms in the Fourier series, there are an infinite number of spatial harmonics.
• Only a few of the spatial harmonics are propagating waves. Only these can carry energy away from a device.

Plane Wave Spectrum from Crossed Gratings
Doubly-periodic gratings, also called crossed gratings, can diffract waves into many directions. They are described by two grating vectors, $\mathbf{K}_x$ and $\mathbf{K}_y$.

Two boundary conditions are necessary here.

$$k_x (m) = k_{x, \text{inc}} - mK_x \quad m = \ldots, -2, 0, 1, 2, \ldots$$

$$k_y (n) = k_{y, \text{inc}} - nK_y \quad n = \ldots, -2, 0, 1, 2, \ldots$$

$$\vec{K}_x = \frac{2\pi}{\Lambda_x} \hat{x}$$

$$\vec{K}_y = \frac{2\pi}{\Lambda_y} \hat{y}$$

**Diffraction from Crossed Gratings**
Visualizing the Transverse Wave Vector Expansion

$$k_x(m)$$

$$k_y(n)$$

$$k_{\text{tran}}(m,n) = k_x(m) \hat{x} + k_y(n) \hat{y}$$

Longitudinal Wave Vector Expansion

The longitudinal components of the wave vectors are computed as

$$k_z^{\text{ref}}(m,n) = \sqrt{(k_z^{\text{ref}})^2 - k_x^2(m) - k_y^2(n)}$$

$$k_z^{\text{im}}(m,n) = \sqrt{(k_z^{\text{im}})^2 - k_x^2(m) - k_y^2(n)}$$

The center few modes will have real $$k_z$$'s. These correspond to propagating waves. The others will have imaginary $$k_z$$'s and correspond to evanescent waves that do not transport energy.
Visualizing the Overall Wave Vector Expansion

The Grating Spectrometer
What is a Grating Spectrometer

Diffraction Grating

Separated Colors

Input Light

Spectral Sensitivity

We start with the grating equation.

\[ n_{\text{avg}} \sin \left( \theta(m) \right) = n_{\text{inc}} \sin \theta_{\text{inc}} - m \frac{\lambda_0}{\Lambda_x} \]

We define spectral sensitivity as how much the diffracted angle changes with respect to wavelength \( \frac{\partial \theta(m)}{\partial \lambda_0} \).

\[ \frac{\partial \theta(m)}{\partial \lambda_0} = - \frac{m}{\Lambda_x n_{\text{avg}} \cos \theta_m} \]

\[ \Delta \theta(m) \approx \frac{m}{\Lambda_x n_{\text{avg}} \cos \left[ \theta(m) \right]} \Delta \lambda_0 \]

This equation tells us how to maximize sensitivity.

1. Diffract into higher order modes (\( \uparrow m \)).
2. Use short period gratings (\( \downarrow \Lambda_x \)).
3. Diffract into large angles (\( \uparrow \theta(m) \)).
4. Diffract into air (\( \downarrow n_{\text{avg}} \)).
Littrow Gratings
Littrow Configuration

In the littrow configuration, the +1-order reflected mode is parallel to the incident wave vector. This forms a spectrally selective mirror.

Conditions for the Littrow Configuration

The grating equation is

\[ n \sin[\theta(m)] = n \sin \theta_{inc} - m \frac{\lambda_0}{\Lambda_x} \]

The littrow configuration occurs when

\[ \theta(+1) = -\theta_{inc} \]

The condition for the littrow configuration is found by substituting this into the grating equation.

\[ 2n \sin \theta_{inc} = \frac{\lambda_0}{\Lambda_x} \]
Typically only a cone of angles $\Delta \theta$ reflected from a grating is detected.

We wish to find $d\lambda/d\theta$ by differentiating our last equation.

\[ \frac{d\lambda}{d\theta} = 2n\Lambda_x \cos \theta \]

Typically this is used to calculate the reflected bandwidth.

\[ \Delta \lambda = 2n\Lambda_x \cos \theta \cdot \Delta \theta \] \text{Linewidth (optics and photonics)}

\[ \Delta f = \frac{2n\Lambda_x f^2 \cos \theta}{c_0} \Delta \theta \] \text{Bandwidth (RF and microwave)}

**Example (1 of 2)**

Design a metallic grating in air that is to be operated in the littrow configuration at 10 GHz at an angle of 45°.

**Solution**
Right away, we know that

\[ n = 1.0 \]
\[ \theta_{\text{inc}} = 45^\circ \]
\[ \lambda_0 = \frac{c_0}{f} = \frac{3 \times 10^8 \text{ m/s}}{10 \text{ GHz}} = 3.00 \text{ cm} \]

The grating period is then found to be

\[ \Lambda_x = \frac{\lambda_0}{2n \sin \theta_{\text{inc}}} = \frac{3.00 \text{ cm}}{2(1.0) \sin(45^\circ)} = 2.12 \text{ cm} \]
Example (2 of 2)

Solution continued
Assuming a 5° cone of angles is detected upon reflection, the bandwidth is

\[ \Delta f = \frac{2(1.0)(2.12 \text{ cm})(10 \text{ GHz})^2 \cos(45°)}{3 \times 10^8 \frac{\pi}{180°} \frac{\pi}{5°}} = 0.87 \text{ GHz} \]

Patterned Fanout Gratings
Near-Field to Far-Field

After propagating a long distance, the field within a plane tends toward the Fourier transform of the initial field.

\[ \tilde{E}(x, y, 0) \rightarrow \tilde{E}(x, y, L) \]

What is a Patterned Fanout Grating?

Diffraction grating forces the field to take on the profile of the inverse Fourier transform of an image. After propagating very far, the field takes on the profile of the image.
Gerchberg-Saxton Algorithm:

**Initialization**

1. **Far-Field**
   - Step 1: Start with desired far-field image.

2. **Near-Field**
   - Step 2: Calculate near-field amplitude phase.
   - Step 3: Replace amplitude with desired image.

3. **Far-Field**
   - Step 4: Calculate far-field amplitude phase.

4. **Near-Field**
   - Step 5: Replace amplitude with desired far-field image.

5. **Far-Field**
   - Step 6: Calculate far-field amplitude phase.

6. **Near-Field**
   - Step 7: Replace amplitude with desired near-field image.

Gerchberg-Saxton Algorithm:

**Iteration**

1. **Far-Field**
   - Step 4: Calculate far-field amplitude phase.

2. **Near-Field**
   - Step 5: Replace amplitude with desired far-field image.

3. **Far-Field**
   - Step 6: Calculate far-field amplitude phase.

4. **Near-Field**
   - Step 7: Replace amplitude with desired near-field image.

5. **Far-Field**
   - Step 8: Calculate far-field amplitude phase.

6. **Near-Field**
   - Step 9: Replace amplitude with desired near-field image.
Gerchberg-Saxton Algorithm:

End

Near-Field

Far-Field

This is the phase function of the diffractive optical element.

This is what the final image will look like.

After several dozen iterations...

The Final Fanout Grating

A surface relief pattern is etched into glass to induce the phase function onto the beam of light.

We could also print an amplitude mask using a high resolution laser printer.
Diffractive Optical Elements

What is a Diffractive Optical Element

If the device is only required to operate over a narrow band, devices can be “flattened.”

The flattened device is called a diffractive optical element (DOE).