

EE 5303  
**Electromagnetic Analysis Using  
Finite-Difference Time-Domain**

Lecture #14

# 3D Update Equations with Uniaxial PML

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## Lecture Outline



- Review of Lecture 13
- Conversion to the time-domain
- Numerical approximations
- Derivation of the update equations
- Summary of all update equations

# Review of Lecture 13

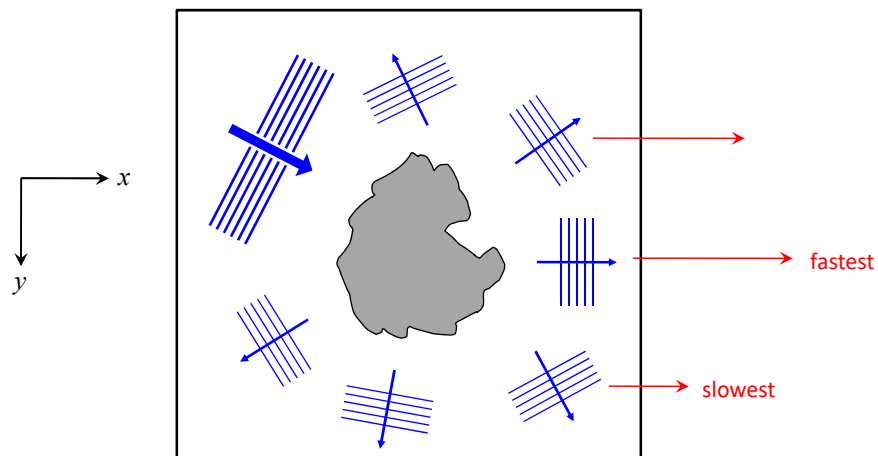
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## No PAB in Two Dimensions

TDT

If we model a wave hitting some device or object, it will scatter the applied wave into potentially many directions. Waves at different angles travel at different speeds through a boundary. Therefore, the PAB condition can only be satisfied for one direction, not all directions.

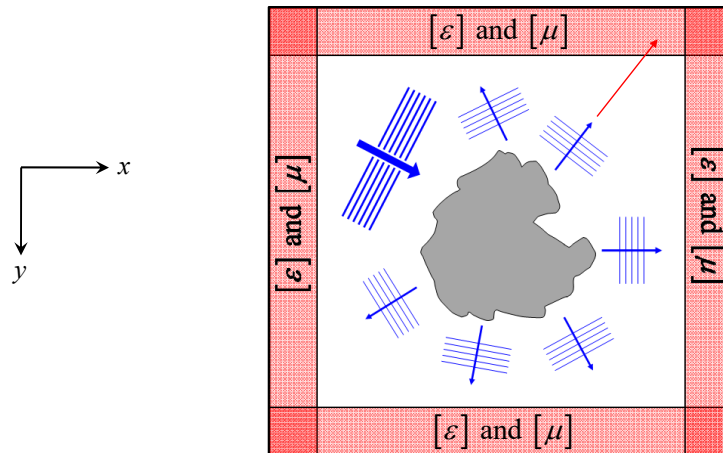


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## Anisotropy to the Rescue!!

It turns out we can prevent reflections at all angles, all frequencies, and for all polarizations if we allow our absorbing material to be diagonally anisotropic.



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## The PML Parameters (1 of 2)

We need to generalize our PML for waves incident on all boundaries.

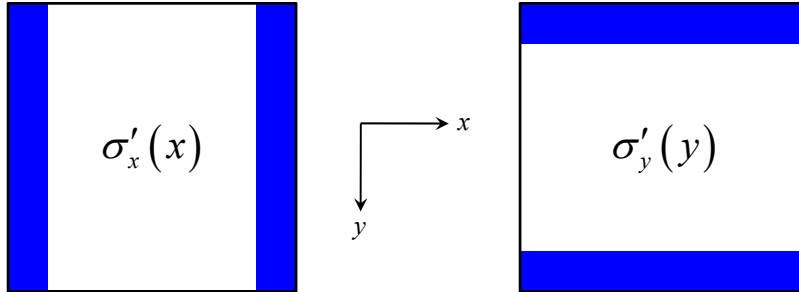
$$[S] = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix} \quad \begin{aligned} s_x(x) &= 1 + \frac{\sigma'_x(x)}{j\omega\epsilon_0} \\ s_y(y) &= 1 + \frac{\sigma'_y(y)}{j\omega\epsilon_0} \\ s_z(z) &= 1 + \frac{\sigma'_z(z)}{j\omega\epsilon_0} \end{aligned}$$

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# The PML Parameters (2 of 2) FDTD

The loss terms should increase gradually into the PMLs.



$$\sigma'_x(x) = \frac{\epsilon_0}{2\Delta t} \left(\frac{x}{L_x}\right)^3 \quad \sigma'_y(y) = \frac{\epsilon_0}{2\Delta t} \left(\frac{y}{L_y}\right)^3 \quad \sigma'_z(z) = \frac{\epsilon_0}{2\Delta t} \left(\frac{z}{L_z}\right)^3$$

$L_\gamma \equiv$  length of the PML

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# Maxwell's Equations with PML FDTD

$$\nabla \times \tilde{\vec{E}}(\omega) = -j\omega \frac{[\mu_r]}{c_0} [\mathcal{S}] \tilde{\vec{H}}(\omega) \Rightarrow \begin{aligned} j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) H_x(\omega) &= -\frac{c_0}{\mu_x} \left[ \frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z} \right] \\ j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) H_y(\omega) &= -\frac{c_0}{\mu_y} \left[ \frac{\partial \tilde{E}_x(\omega)}{\partial z} - \frac{\partial \tilde{E}_z(\omega)}{\partial x} \right] \\ j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1} H_z(\omega) &= -\frac{c_0}{\mu_z} \left[ \frac{\partial \tilde{E}_y(\omega)}{\partial x} - \frac{\partial \tilde{E}_x(\omega)}{\partial y} \right] \end{aligned}$$

$$\nabla \times \tilde{\vec{H}}(\omega) = \eta_0 \sigma \tilde{\vec{E}}(\omega) + \frac{j\omega}{c_0} [\mathcal{S}] \tilde{\vec{D}}(\omega) \Rightarrow \begin{aligned} j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) \tilde{D}_x(\omega) &= c_0 \left[ \frac{\partial H_z(\omega)}{\partial y} - \frac{\partial H_y(\omega)}{\partial z} \right] - \frac{\sigma_{xx}}{\epsilon_0} \tilde{E}_x(\omega) \\ j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) \tilde{D}_y(\omega) &= c_0 \left[ \frac{\partial H_x(\omega)}{\partial z} - \frac{\partial H_z(\omega)}{\partial x} \right] - \frac{\sigma_{yy}}{\epsilon_0} \tilde{E}_y(\omega) \\ j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1} \tilde{D}_z(\omega) &= c_0 \left[ \frac{\partial H_y(\omega)}{\partial x} - \frac{\partial H_x(\omega)}{\partial y} \right] - \frac{\sigma_{zz}}{\epsilon_0} \tilde{E}_z(\omega) \end{aligned}$$

$$\tilde{\vec{D}}(\omega) = [\epsilon_r] \tilde{\vec{E}}(\omega) \Rightarrow \begin{aligned} \tilde{D}_x(\omega) &= \epsilon_{xx} \tilde{E}_x(\omega) \\ \tilde{D}_y(\omega) &= \epsilon_{yy} \tilde{E}_y(\omega) \\ \tilde{D}_z(\omega) &= \epsilon_{zz} \tilde{E}_z(\omega) \end{aligned}$$

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# Conversion to the Time-Domain

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## Assume No Loss ( $\sigma=0$ )

TD

$$\nabla \times \tilde{\mathbf{E}}(\omega) = -j\omega \frac{[\mu_r]}{c_0} [\mathbf{s}] \tilde{\mathbf{H}}(\omega) \Rightarrow \begin{aligned} j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) H_x(\omega) &= -\frac{c_0}{\mu_x} \left[ \frac{\partial \tilde{E}_z(\omega)}{\partial y} - \frac{\partial \tilde{E}_y(\omega)}{\partial z} \right] \\ j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) H_y(\omega) &= -\frac{c_0}{\mu_y} \left[ \frac{\partial \tilde{E}_x(\omega)}{\partial z} - \frac{\partial \tilde{E}_z(\omega)}{\partial x} \right] \\ j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1} H_z(\omega) &= -\frac{c_0}{\mu_z} \left[ \frac{\partial \tilde{E}_y(\omega)}{\partial x} - \frac{\partial \tilde{E}_x(\omega)}{\partial y} \right] \end{aligned}$$

$$\nabla \times \tilde{\mathbf{H}}(\omega) = \eta_0 \sigma \tilde{\mathbf{E}}(\omega) + \frac{j\omega}{c_0} [\mathbf{s}] \tilde{\mathbf{D}}(\omega) \Rightarrow \begin{aligned} j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) \tilde{D}_x(\omega) &= c_0 \left[ \frac{\partial H_z(\omega)}{\partial y} - \frac{\partial H_y(\omega)}{\partial z} \right] \\ j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) \tilde{D}_y(\omega) &= c_0 \left[ \frac{\partial H_x(\omega)}{\partial z} - \frac{\partial H_z(\omega)}{\partial x} \right] \\ j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right)^{-1} \tilde{D}_z(\omega) &= c_0 \left[ \frac{\partial H_y(\omega)}{\partial x} - \frac{\partial H_x(\omega)}{\partial y} \right] \end{aligned}$$

$$\tilde{\mathbf{D}}(\omega) = [\epsilon_r] \tilde{\mathbf{E}}(\omega) \Rightarrow \begin{aligned} \tilde{D}_x(\omega) &= \epsilon_x \tilde{E}_x(\omega) \\ \tilde{D}_y(\omega) &= \epsilon_y \tilde{E}_y(\omega) \\ \tilde{D}_z(\omega) &= \epsilon_z \tilde{E}_z(\omega) \end{aligned}$$

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## Recall Fourier Transform Properties

$$\mathbf{F}\{g(t)\} = G(\omega)$$

$$\mathbf{F}\{ag(t)\} = aG(\omega)$$

$$\mathbf{F}\left\{\frac{d^a}{dt^a}g(t)\right\} = (j\omega)^a G(\omega)$$

$$\mathbf{F}\left\{\int_{-\infty}^t g(\tau)d\tau\right\} = \frac{1}{j\omega}G(\omega)$$

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## Prepare Maxwell's Equations for Conversion (1 of 2)

We will start with this equation. The procedure for all other equations has the same basic steps.

$$j\omega\left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1}\left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right)H_x(\omega) = -\frac{c_0}{\mu_{xx}}\left[\frac{\partial\tilde{E}_z(\omega)}{\partial y} - \frac{\partial\tilde{E}_y(\omega)}{\partial z}\right]$$

First, let

$$C_x^E(\omega) = \frac{\partial\tilde{E}_z(\omega)}{\partial y} - \frac{\partial\tilde{E}_y(\omega)}{\partial z}$$

The equation now becomes

$$j\omega\left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1}\left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right)\left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right)H_x(\omega) = -\frac{c_0}{\mu_{xx}}C_x^E(\omega)$$

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## Prepare Maxwell's Equations for Conversion (2 of 2)

The inverse term on the left is brought to the right.

$$j\omega \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right)^{-1} \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) H_x(\omega) = -\frac{c_0}{\mu_{xx}} C_x^E(\omega)$$

$$\downarrow$$

$$j\omega \left(1 + \frac{\sigma'_y}{j\omega\epsilon_0}\right) \left(1 + \frac{\sigma'_z}{j\omega\epsilon_0}\right) H_x(\omega) = -\frac{c_0}{\mu_{xx}} \left(1 + \frac{\sigma'_x}{j\omega\epsilon_0}\right) C_x^E(\omega)$$

The equation is then expanded by multiplying all of the terms.

$$j\omega H_x(\omega) + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(\omega) + \frac{1}{j\omega} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\omega) = -\frac{c_0}{\mu_{xx}} C_x^E(\omega) - \frac{1}{j\omega} \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\omega)$$

For easy conversion to the time-domain, we want all the terms to be multiplied by  $(j\omega)^a$ .

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## Convert Each Term to Time-Domain Individually

$$j\omega H_x(\omega) + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(\omega) + \frac{1}{j\omega} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\omega) = -\frac{c_0}{\mu_{xx}} C_x^E(\omega) - \frac{1}{j\omega} \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\omega)$$

$$j\omega H_x(\omega) \Leftrightarrow \frac{\partial H_x(t)}{\partial t}$$

$$\frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(\omega) \Leftrightarrow \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(t)$$

$$\frac{1}{j\omega} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\omega) \Leftrightarrow \int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau$$

$$\frac{c_0}{\mu_{xx}} C_x^E(\omega) \Leftrightarrow \frac{c_0}{\mu_{xx}} C_x^E(t)$$

$$\frac{1}{j\omega} \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\omega) \Leftrightarrow \int_{-\infty}^t \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\tau) d\tau$$

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## The Time-Domain Equation with PML

FDTD

$$j\omega H_x(\omega) + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(\omega) + \frac{1}{j\omega} \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\omega) = -\frac{c_0}{\mu_{xx}} C_x^E(\omega) - \frac{1}{j\omega} \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\omega)$$

$$\frac{\partial H_x(t)}{\partial t} + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(t) + \int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau = -\frac{c_0}{\mu_{xx}} C_x^E(t) - \int_{-\infty}^t \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\tau) d\tau$$

$$\frac{\partial H_x(t)}{\partial t} + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(t) + \int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau = -\frac{c_0}{\mu_{xx}} C_x^E(t) - \int_{-\infty}^t \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\tau) d\tau$$

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## Summary of All Time-Domain Equations

FDTD

$$\frac{\partial}{\partial t} H_x(t) + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(t) + \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} \int_{-\infty}^t H_x(\tau) d\tau = -\frac{c_0}{\mu_{xx}} \left[ \frac{\partial \tilde{E}_z(t)}{\partial y} - \frac{\partial \tilde{E}_y(t)}{\partial z} \right] - \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} \int_{-\infty}^t \left[ \frac{\partial \tilde{E}_z(\tau)}{\partial y} - \frac{\partial \tilde{E}_y(\tau)}{\partial z} \right] d\tau$$

$$\frac{\partial}{\partial t} H_y(t) + \frac{\sigma'_x + \sigma'_z}{\epsilon_0} H_y(t) + \frac{\sigma'_x \sigma'_z}{\epsilon_0^2} \int_{-\infty}^t H_y(\tau) d\tau = -\frac{c_0}{\mu_{yy}} \left[ \frac{\partial \tilde{E}_x(t)}{\partial z} - \frac{\partial \tilde{E}_z(t)}{\partial x} \right] - \frac{c_0 \sigma'_y}{\epsilon_0 \mu_{yy}} \int_{-\infty}^t \left[ \frac{\partial \tilde{E}_x(\tau)}{\partial z} - \frac{\partial \tilde{E}_z(\tau)}{\partial x} \right] d\tau$$

$$\frac{\partial}{\partial t} H_z(t) + \frac{\sigma'_x + \sigma'_y}{\epsilon_0} H_z(t) + \frac{\sigma'_x \sigma'_y}{\epsilon_0^2} \int_{-\infty}^t H_z(\tau) d\tau = -\frac{c_0}{\mu_{zz}} \left[ \frac{\partial \tilde{E}_y(t)}{\partial x} - \frac{\partial \tilde{E}_x(t)}{\partial y} \right] - \frac{c_0 \sigma'_z}{\epsilon_0 \mu_{zz}} \int_{-\infty}^t \left[ \frac{\partial \tilde{E}_y(\tau)}{\partial x} - \frac{\partial \tilde{E}_x(\tau)}{\partial y} \right] d\tau$$

$$\frac{\partial}{\partial t} \tilde{D}_x(t) + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} \tilde{D}_x(t) + \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} \int_{-\infty}^t \tilde{D}_x(\tau) d\tau = c_0 \left[ \frac{\partial H_z(t)}{\partial y} - \frac{\partial H_y(t)}{\partial z} \right] + \frac{c_0 \sigma'_x}{\epsilon_0} \int_{-\infty}^t \left[ \frac{\partial H_z(\tau)}{\partial y} - \frac{\partial H_y(\tau)}{\partial z} \right] d\tau$$

$$\frac{\partial}{\partial t} \tilde{D}_y(t) + \frac{\sigma'_x + \sigma'_z}{\epsilon_0} \tilde{D}_y(t) + \frac{\sigma'_x \sigma'_z}{\epsilon_0^2} \int_{-\infty}^t \tilde{D}_y(\tau) d\tau = c_0 \left[ \frac{\partial H_x(t)}{\partial z} - \frac{\partial H_z(t)}{\partial x} \right] + \frac{c_0 \sigma'_y}{\epsilon_0} \int_{-\infty}^t \left[ \frac{\partial H_x(\tau)}{\partial z} - \frac{\partial H_z(\tau)}{\partial x} \right] d\tau$$

$$\frac{\partial}{\partial t} \tilde{D}_z(t) + \frac{\sigma'_x + \sigma'_y}{\epsilon_0} \tilde{D}_z(t) + \frac{\sigma'_x \sigma'_y}{\epsilon_0^2} \int_{-\infty}^t \tilde{D}_z(\tau) d\tau = c_0 \left[ \frac{\partial H_y(t)}{\partial x} - \frac{\partial H_x(t)}{\partial y} \right] + \frac{c_0 \sigma'_z}{\epsilon_0} \int_{-\infty}^t \left[ \frac{\partial H_y(\tau)}{\partial x} - \frac{\partial H_x(\tau)}{\partial y} \right] d\tau$$

$$\tilde{D}_x(t) = \epsilon_{xx} \tilde{E}_x(t)$$

$$\tilde{D}_y(t) = \epsilon_{yy} \tilde{E}_y(t)$$

$$\tilde{D}_z(t) = \epsilon_{zz} \tilde{E}_z(t)$$

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# Numerical Approximations

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## We Will Handle Each Term Separately

The first governing equation in the time-domain was

$$\frac{\partial H_x(t)}{\partial t} + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(t) + \int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau = -\frac{c_0}{\mu_{xx}} C_x^E(t) - \int_{-\infty}^t \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\tau) d\tau$$

The terms listed separately are:

Term 1:  $\frac{\partial H_x(t)}{\partial t}$

Term 2:  $\frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(t)$

Term 3:  $\int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau$

Term 4:  $-\frac{c_0}{\mu_{xx}} C_x^E(t)$

Term 5:  $-\int_{-\infty}^t \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\tau) d\tau$

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## Term 1



We previously approximated this term using finite-differences as follows.

$$\frac{\partial H_x(t)}{\partial t} \cong \frac{H_x|_{t+\Delta t/2}^{i,j,k} - H_x|_{t-\Delta t/2}^{i,j,k}}{\Delta t}$$

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## Term 2



The trick here is that we need to interpolate the  $H$  field at time  $t$ . We do this by averaging the values at time  $t+\Delta t/2$  and  $t-\Delta t/2$ .

$$\frac{\sigma_y^H + \sigma_z^H}{\epsilon_0} H_x(t) \cong \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{\epsilon_0} \frac{H_x|_{t+\Delta t/2}^{i,j,k} + H_x|_{t-\Delta t/2}^{i,j,k}}{2}$$

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### Term 3 (1 of 3)

We approximate the integral in this equation with a summation.

$$\int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau = \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} \int_{-\infty}^t H_x(\tau) d\tau$$

$$\cong \frac{\sigma_y^H \sigma_z^H}{\epsilon_0^2} \sum_{T=\Delta t/2}^{t+\Delta t/2} H_x|_T^{i,j,k} \Delta t$$

We have a problem. The summation includes up to  $t+\Delta t/2$ , but the integral only goes up to  $t$ . We are accidentally integrating too far. This happened because the summation is integrating a term that exists at the half time steps.

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### Term 3 (2 of 3)

We pull out the last term in the summation.

$$\int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau \cong \frac{\sigma_y^H \sigma_z^H}{\epsilon_0^2} \sum_{T=\Delta t/2}^{t+\Delta t/2} H_x|_T^{i,j,k} \Delta t$$

$$\cong \frac{\sigma_y^H \sigma_z^H}{\epsilon_0^2} \left( H_x|_{t+\Delta t/2}^{i,j,k} \frac{\Delta t}{2} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x|_T^{i,j,k} \Delta t \right)$$

We force the extracted term to only integrate over half of a time step.

$$\int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau \cong \frac{\sigma_y^H \sigma_z^H}{\epsilon_0^2} \left( \underbrace{\frac{H_x|_{t+\Delta t/2}^{i,j,k} + H_x|_{t-\Delta t/2}^{i,j,k}}{2}}_{H \text{ field interpolated at time } t} \frac{\Delta t}{2} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x|_T^{i,j,k} \Delta t \right)$$

Half time step.

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### Term 3 (3 of 3)

We factor out the  $\Delta t$  term.

$$\int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau \cong \frac{\sigma_y^H \sigma_z^H \Delta t}{\epsilon_0^2} \left( \frac{H_x|_{t-\Delta t/2}^{i,j,k} + H_x|_{t+\Delta t/2}^{i,j,k}}{4} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x|_T^{i,j,k} \right)$$



The end-time on this summation implies that we will update the summation before updating the H field.

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### Term 4

We approximated this term using finite-differences in a previous lecture.

$$-\frac{c_0}{\mu_{xx}} C_x^E(t) \cong -\frac{c_0}{\mu_{xx}|_{i,j,k}} C_x^E|_t^{i,j,k}$$

Recall that the curl term is

$$C_x^E|_t^{i,j,k} = \frac{\tilde{E}_z|_t^{i,j+1,k} - \tilde{E}_z|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y|_t^{i,j,k+1} - \tilde{E}_y|_t^{i,j,k}}{\Delta z}$$

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## Term 5

We approximate the integral in this equation with a summation.

$$\begin{aligned}
 -\int_{-\infty}^t \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\tau) d\tau &= -\frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} \int_{-\infty}^t C_x^E(\tau) d\tau \\
 &\cong -\frac{c_0 \sigma_x^H|^{i,j,k}}{\epsilon_0 \mu_{xx}|^{i,j,k}} \sum_{T=0}^t C_x^E|_T^{i,j,k} \Delta t \\
 &\cong -\frac{c_0 \Delta t \sigma_x^H|^{i,j,k}}{\epsilon_0 \mu_{xx}|^{i,j,k}} \sum_{T=0}^t C_x^E|_T^{i,j,k}
 \end{aligned}$$

The end-time on this summation implies that we will update the summation before updating the E field.



There are no problems with this summation because it is integrating a term that exists at integer time steps.

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## Putting it All Together

Putting all the terms together, the equation is

$$\frac{\partial H_x(t)}{\partial t} + \frac{\sigma'_y + \sigma'_z}{\epsilon_0} H_x(t) + \int_{-\infty}^t \frac{\sigma'_y \sigma'_z}{\epsilon_0^2} H_x(\tau) d\tau = -\frac{c_0}{\mu_{xx}} C_x^E(t) - \int_{-\infty}^t \frac{c_0 \sigma'_x}{\epsilon_0 \mu_{xx}} C_x^E(\tau) d\tau$$



$$\begin{aligned}
 &\frac{H_x|_{t+\Delta t/2}^{i,j,k} - H_x|_{t-\Delta t/2}^{i,j,k}}{\Delta t} + \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{\epsilon_0} \frac{H_x|_{t+\Delta t/2}^{i,j,k} + H_x|_{t-\Delta t/2}^{i,j,k}}{2} + \frac{\sigma_y^H \sigma_z^H \Delta t}{\epsilon_0^2} \left( \frac{H_x|_{t-\Delta t/2}^{i,j,k} + H_x|_{t+\Delta t/2}^{i,j,k}}{4} + \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x|_T^{i,j,k} \right) \\
 &= -\frac{c_0}{\mu_{xx}|^{i,j,k}} C_x^E|_t^{i,j,k} - \frac{c_0 \Delta t \sigma_x^H|^{i,j,k}}{\epsilon_0 \mu_{xx}|^{i,j,k}} \sum_{T=0}^t C_x^E|_T^{i,j,k}
 \end{aligned}$$

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## Summary of All Numerical Equations

$$\frac{H_x^{n,j,k} - H_x^{n-M/2,j,k} + \sigma_x^H |^{n,j,k} + \sigma_z^H |^{n,j,k} \left( \frac{H_x^{n,j,k} + H_x^{n-M/2,j,k}}{2} \right) + \left( \frac{\sigma_y^H |^{n,j,k}}{\epsilon_0^2} \right) \left( \frac{\sigma_z^H |^{n,j,k}}{\epsilon_0^2} \right) \Delta t \left[ \frac{1}{4} \left( H_x^{n,j,k} + H_x^{n-M/2,j,k} \right) + \sum_{r=0}^{n-M/2} H_x^{r,j,k} \right]}{\Delta t} = -\frac{c_0}{\mu_{xx} |^{n,j,k}} C_x^E |^{n,j,k} - \frac{c_0 \Delta t \sigma_z^H |^{n,j,k}}{\epsilon_0 \mu_{xx} |^{n,j,k}} \sum_{r=0}^l C_x^E |^{r,j,k}$$

$$\frac{H_y^{n,j,k} - H_y^{n-M/2,j,k} + \sigma_x^H |^{n,j,k} + \sigma_z^H |^{n,j,k} \left( \frac{H_y^{n,j,k} + H_y^{n-M/2,j,k}}{2} \right) + \left( \frac{\sigma_x^H |^{n,j,k}}{\epsilon_0^2} \right) \left( \frac{\sigma_z^H |^{n,j,k}}{\epsilon_0^2} \right) \Delta t \left[ \frac{1}{4} \left( H_y^{n,j,k} + H_y^{n-M/2,j,k} \right) + \sum_{r=0}^{n-M/2} H_y^{r,j,k} \right]}{\Delta t} = -\frac{c_0}{\mu_{yy} |^{n,j,k}} C_y^E |^{n,j,k} - \frac{c_0 \Delta t \sigma_x^H |^{n,j,k}}{\epsilon_0 \mu_{yy} |^{n,j,k}} \sum_{r=0}^l C_y^E |^{r,j,k}$$

$$\frac{H_z^{n,j,k} - H_z^{n-M/2,j,k} + \sigma_x^H |^{n,j,k} + \sigma_y^H |^{n,j,k} \left( \frac{H_z^{n,j,k} + H_z^{n-M/2,j,k}}{2} \right) + \left( \frac{\sigma_x^H |^{n,j,k}}{\epsilon_0^2} \right) \left( \frac{\sigma_y^H |^{n,j,k}}{\epsilon_0^2} \right) \Delta t \left[ \frac{1}{4} \left( H_z^{n,j,k} + H_z^{n-M/2,j,k} \right) + \sum_{r=0}^{n-M/2} H_z^{r,j,k} \right]}{\Delta t} = -\frac{c_0}{\mu_{zz} |^{n,j,k}} C_z^E |^{n,j,k} - \frac{c_0 \Delta t \sigma_z^H |^{n,j,k}}{\epsilon_0 \mu_{zz} |^{n,j,k}} \sum_{r=0}^l C_z^E |^{r,j,k}$$
  

$$\frac{\bar{D}_x^{n,j,k} - \bar{D}_x^{n-M/2,j,k} + \sigma_x^D |^{n,j,k} + \sigma_z^D |^{n,j,k} \left( \frac{\bar{D}_x^{n,j,k} + \bar{D}_x^{n-M/2,j,k}}{2} \right) + \left( \frac{\sigma_y^D |^{n,j,k}}{\epsilon_0^2} \right) \left( \frac{\sigma_z^D |^{n,j,k}}{\epsilon_0^2} \right) \Delta t \left[ \frac{1}{4} \left( \bar{D}_x^{n,j,k} + \bar{D}_x^{n-M/2,j,k} \right) + \sum_{r=0}^{n-M/2} \bar{D}_x^{r,j,k} \right]}{\Delta t} = c_0 C_x^H |^{n,j,k} + \frac{c_0 \Delta t \sigma_x^D |^{n,j,k}}{\epsilon_0} \sum_{r=0}^{n-M/2} C_x^H |^{r,j,k}$$

$$\frac{\bar{D}_y^{n,j,k} - \bar{D}_y^{n-M/2,j,k} + \sigma_x^D |^{n,j,k} + \sigma_z^D |^{n,j,k} \left( \frac{\bar{D}_y^{n,j,k} + \bar{D}_y^{n-M/2,j,k}}{2} \right) + \left( \frac{\sigma_x^D |^{n,j,k}}{\epsilon_0^2} \right) \left( \frac{\sigma_z^D |^{n,j,k}}{\epsilon_0^2} \right) \Delta t \left[ \frac{1}{4} \left( \bar{D}_y^{n,j,k} + \bar{D}_y^{n-M/2,j,k} \right) + \sum_{r=0}^{n-M/2} \bar{D}_y^{r,j,k} \right]}{\Delta t} = c_0 C_y^H |^{n,j,k} + \frac{c_0 \Delta t \sigma_y^D |^{n,j,k}}{\epsilon_0} \sum_{r=0}^{n-M/2} C_y^H |^{r,j,k}$$

$$\frac{\bar{D}_z^{n,j,k} - \bar{D}_z^{n-M/2,j,k} + \sigma_x^D |^{n,j,k} + \sigma_y^D |^{n,j,k} \left( \frac{\bar{D}_z^{n,j,k} + \bar{D}_z^{n-M/2,j,k}}{2} \right) + \left( \frac{\sigma_x^D |^{n,j,k}}{\epsilon_0^2} \right) \left( \frac{\sigma_y^D |^{n,j,k}}{\epsilon_0^2} \right) \Delta t \left[ \frac{1}{4} \left( \bar{D}_z^{n,j,k} + \bar{D}_z^{n-M/2,j,k} \right) + \sum_{r=0}^{n-M/2} \bar{D}_z^{r,j,k} \right]}{\Delta t} = c_0 C_z^H |^{n,j,k} + \frac{c_0 \Delta t \sigma_z^D |^{n,j,k}}{\epsilon_0} \sum_{r=0}^{n-M/2} C_z^H |^{r,j,k}$$
  

$$\bar{D}_x^{n,j,k} = \left( \epsilon_{xx} |^{n,j,k} \right) \bar{E}_x^{n,j,k}$$

$$\bar{D}_y^{n,j,k} = \left( \epsilon_{yy} |^{n,j,k} \right) \bar{E}_y^{n,j,k}$$

$$\bar{D}_z^{n,j,k} = \left( \epsilon_{zz} |^{n,j,k} \right) \bar{E}_z^{n,j,k}$$

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# Derivation of the Update Equations

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## The Starting Point

Just as before, we solve the numerical equation for the future time value of  $H$ . This term appears three times and these must be collected.

$$\begin{aligned} & \frac{H_x^{i,j,k}|_{t+\Delta t/2} - H_x^{i,j,k}|_{t-\Delta t/2}}{\Delta t} + \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{\varepsilon_0} \left( \frac{H_x^{i,j,k}|_{t+\Delta t/2} + H_x^{i,j,k}|_{t-\Delta t/2}}{2} \right) \\ & + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{\varepsilon_0^2} \left[ \frac{1}{4} (H_x^{i,j,k}|_{t+\Delta t/2} + H_x^{i,j,k}|_{t-\Delta t/2}) + \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x^{i,j,k}|_T \right] \\ & = -\frac{c_0}{\mu_{xx}|^{i,j,k}} C_x^E|_t^{i,j,k} - \frac{c_0\Delta t}{\varepsilon_0} \frac{\sigma_x^H|^{i,j,k}}{\mu_{xx}|^{i,j,k}} \sum_{T=0}^t C_x^E|_T^{i,j,k} \end{aligned}$$

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## Expand the Equation

We start by expanding the equation by multiplying all of the terms.

$$\begin{aligned} & \frac{H_x^{i,j,k}|_{t+\Delta t/2} - H_x^{i,j,k}|_{t-\Delta t/2}}{\Delta t} + \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{\varepsilon_0} \left( \frac{H_x^{i,j,k}|_{t+\Delta t/2} + H_x^{i,j,k}|_{t-\Delta t/2}}{2} \right) + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{\varepsilon_0^2} \left[ \frac{1}{4} (H_x^{i,j,k}|_{t+\Delta t/2} + H_x^{i,j,k}|_{t-\Delta t/2}) + \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x^{i,j,k}|_T \right] \\ & = -\frac{c_0}{\mu_{xx}|^{i,j,k}} C_x^E|_t^{i,j,k} - \frac{c_0\Delta t}{\varepsilon_0} \frac{\sigma_x^H|^{i,j,k}}{\mu_{xx}|^{i,j,k}} \sum_{T=0}^t C_x^E|_T^{i,j,k} \end{aligned}$$



$$\begin{aligned} & \frac{1}{\Delta t} H_x^{i,j,k}|_{t+\Delta t/2} - \frac{1}{\Delta t} H_x^{i,j,k}|_{t-\Delta t/2} + \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\varepsilon_0} \right) H_x^{i,j,k}|_{t+\Delta t/2} + \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\varepsilon_0} \right) H_x^{i,j,k}|_{t-\Delta t/2} \\ & + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\varepsilon_0^2} H_x^{i,j,k}|_{t+\Delta t/2} + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\varepsilon_0^2} H_x^{i,j,k}|_{t-\Delta t/2} + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{\varepsilon_0^2} \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x^{i,j,k}|_T \\ & = -\frac{c_0}{\mu_{xx}|^{i,j,k}} C_x^E|_t^{i,j,k} - \frac{c_0\Delta t}{\varepsilon_0} \frac{\sigma_x^H|^{i,j,k}}{\mu_{xx}|^{i,j,k}} \sum_{T=0}^t C_x^E|_T^{i,j,k} \end{aligned}$$

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## Collect Common Terms



Next we collect the coefficients of the common field terms.

$$\begin{aligned} & \frac{1}{\Delta t} H_x|_{t+\Delta t/2} - \frac{1}{\Delta t} H_x|_{t-\Delta t/2} + \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\epsilon_0} \right) H_x|_{t+\Delta t/2} + \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\epsilon_0} \right) H_x|_{t-\Delta t/2} \\ & + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\epsilon_0^2} H_x|_{t+\Delta t/2} + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\epsilon_0^2} H_x|_{t-\Delta t/2} + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{\epsilon_0^2} \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x|_T \\ & = -\frac{c_0}{\mu_{xx}|^{i,j,k}} C_x^E|_t + \frac{c_0\Delta t}{\epsilon_0} \frac{\sigma_x^H|^{i,j,k}}{\mu_{xx}|^{i,j,k}} \sum_{T=0}^t C_x^E|_T \end{aligned}$$



$$\begin{aligned} & \left[ \frac{1}{\Delta t} + \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\epsilon_0} \right) + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\epsilon_0^2} \right] H_x|_{t+\Delta t/2} \\ & - \left[ \frac{1}{\Delta t} - \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\epsilon_0} \right) - \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\epsilon_0^2} \right] H_x|_{t-\Delta t/2} \\ & = \left[ -\frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{\epsilon_0^2} \right] \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x|_T + \left[ -\frac{c_0}{\mu_{xx}|^{i,j,k}} \right] C_x^E|_t + \left[ -\frac{c_0\Delta t}{\epsilon_0} \frac{\sigma_x^H|^{i,j,k}}{\mu_{xx}|^{i,j,k}} \right] \sum_{T=0}^t C_x^E|_T \end{aligned}$$

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## Solve for the Future H Field



Solving our numerical equation for the H field at  $t+\Delta t/2$  yields

$$\begin{aligned} H_x|_{t+\Delta t/2} &= \left[ \frac{1}{\Delta t} - \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\epsilon_0} \right) - \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\epsilon_0^2} \right] H_x|_{t-\Delta t/2} + \left[ -\frac{c_0/\mu_{xx}|^{i,j,k}}{\frac{1}{\Delta t} + \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\epsilon_0} \right) + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\epsilon_0^2}} \right] C_x^E|_t \\ & + \left[ -\frac{c_0\Delta t}{\epsilon_0} \frac{\sigma_x^H|^{i,j,k}}{\mu_{xx}|^{i,j,k}} \right] \sum_{T=0}^t C_x^E|_T + \left[ \frac{\frac{\Delta t}{\epsilon_0} (\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})}{\frac{1}{\Delta t} + \left( \frac{\sigma_y^H|^{i,j,k} + \sigma_z^H|^{i,j,k}}{2\epsilon_0} \right) + \frac{(\sigma_y^H|^{i,j,k})(\sigma_z^H|^{i,j,k})\Delta t}{4\epsilon_0^2}} \right] \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x|_T \end{aligned}$$

This is the largest equation in the entire course! 😊

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## Final Form of the Update Equation for $H_x$ FDTD

The update coefficients are computed before the main FDTD loop.

$$m_{Hx0}^{i,j,k} = \frac{1}{\Delta t} \left( \frac{\sigma_y^{i,j,k} + \sigma_z^{i,j,k}}{2\epsilon_0} \right) + \frac{(\sigma_y^{i,j,k})(\sigma_z^{i,j,k})\Delta t}{4\epsilon_0^2} \quad m_{Hx1}^{i,j,k} = \frac{1}{m_{Hx0}^{i,j,k}} \left[ \frac{1}{\Delta t} - \left( \frac{\sigma_y^{i,j,k} + \sigma_z^{i,j,k}}{2\epsilon_0} \right) - \frac{(\sigma_y^{i,j,k})(\sigma_z^{i,j,k})\Delta t}{4\epsilon_0^2} \right]$$

$$m_{Hx2}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{c_0}{\mu_{xx}^{i,j,k}} \quad m_{Hx3}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{c_0 \Delta t \sigma_x^{i,j,k}}{\epsilon_0 \mu_{xx}^{i,j,k}} \quad m_{Hx4}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{\Delta t}{\epsilon_0^2} (\sigma_y^{i,j,k})(\sigma_z^{i,j,k})$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEX}^{i,j,k} = \sum_{T=0}^t C_x^E \Big|_T^{i,j,k} \quad I_{Hx}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\Delta t/2} H_x \Big|_T^{i,j,k} \quad C_x^E \Big|_t^{i,j,k} = \frac{\tilde{E}_z \Big|_t^{i,j,k+1} - \tilde{E}_z \Big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y \Big|_t^{i,j,k+1} - \tilde{E}_y \Big|_t^{i,j,k}}{\Delta z}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_x \Big|_{t+\Delta t/2}^{i,j,k} = (m_{Hx1}^{i,j,k}) H_x \Big|_{t-\Delta t/2}^{i,j,k} + (m_{Hx2}^{i,j,k}) C_x^E \Big|_t^{i,j,k} + (m_{Hx3}^{i,j,k}) I_{CEX} \Big|_t^{i,j,k} + (m_{Hx4}^{i,j,k}) I_{Hx} \Big|_t^{i,j,k}$$

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## Are These Equations Correct? FDTD

A simple way to test these update equations is to set the PML parameters to zero and see if the update equations reduce to what we discussed in a previous lecture.

The update coefficients reduce to

$$m_{Hx1}^{i,j,k} = 1 \quad m_{Hx2}^{i,j,k} = -\frac{c_0 \Delta t}{\mu_{xx}^{i,j,k}} \quad m_{Hx3}^{i,j,k} = 0 \quad m_{Hx4}^{i,j,k} = 0$$

The update equation reduces to

$$H_x \Big|_{t+\Delta t/2}^{i,j,k} = H_x \Big|_{t-\Delta t/2}^{i,j,k} + \left( -\frac{c_0 \Delta t}{\mu_{xx}^{i,j,k}} \right) \left( \frac{\tilde{E}_z \Big|_t^{i,j,k+1} - \tilde{E}_z \Big|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y \Big|_t^{i,j,k+1} - \tilde{E}_y \Big|_t^{i,j,k}}{\Delta z} \right)$$

For problems uniform in the z direction, the z derivative is zero. This equation reduces to exactly what is shown in Lecture 11.

$$H_x \Big|_{t+\Delta t/2}^{i,j} = H_x \Big|_{t-\Delta t/2}^{i,j} + \left( -\frac{c_0 \Delta t}{\mu_{xx}^{i,j}} \right) \left( \frac{\tilde{E}_z \Big|_t^{i,j+1} - \tilde{E}_z \Big|_t^{i,j}}{\Delta y} \right)$$



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# Summary of the Update Equations

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## Final Form of the Update Equation for $H_x$ FDTD

The update coefficients are computed before the main FDTD loop.

$$m_{Hx0}^{i,j,k} = \frac{1}{\Delta t} + \left( \frac{\sigma_y^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\epsilon_0} \right) + \frac{(\sigma_y^{H,i,j,k})(\sigma_z^{H,i,j,k})\Delta t}{4\epsilon_0^2} \quad m_{Hx1}^{i,j,k} = \frac{1}{m_{Hx0}^{i,j,k}} \left[ \frac{1}{\Delta t} - \left( \frac{\sigma_y^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\epsilon_0} \right) - \frac{(\sigma_y^{H,i,j,k})(\sigma_z^{H,i,j,k})\Delta t}{4\epsilon_0^2} \right]$$

$$m_{Hx2}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{c_0}{\mu_{xx}^{i,j,k}} \quad m_{Hx3}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{c_0 \Delta t}{\epsilon_0} \frac{\sigma_x^{H,i,j,k}}{\mu_{xx}^{i,j,k}} \quad m_{Hx4}^{i,j,k} = -\frac{1}{m_{Hx0}^{i,j,k}} \frac{\Delta t}{\epsilon_0^2} (\sigma_y^{H,i,j,k})(\sigma_z^{H,i,j,k})$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEx}^E|_t^{i,j,k} = \sum_{T=0}^t C_x^E|_t^{i,j,k} \quad I_{Hx}^E|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_x|_T^{i,j,k} \quad C_x^E|_t^{i,j,k} = \frac{\tilde{E}_z|_t^{i,j+1,k} - \tilde{E}_z|_t^{i,j,k}}{\Delta y} - \frac{\tilde{E}_y|_t^{i,j,k+1} - \tilde{E}_y|_t^{i,j,k}}{\Delta z}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_x|_{t+\Delta t/2}^{i,j,k} = (m_{Hx1}^{i,j,k}) H_x|_{t-\Delta t/2}^{i,j,k} + (m_{Hx2}^{i,j,k}) C_x^E|_t^{i,j,k} + (m_{Hx3}^{i,j,k}) I_{CEx}^E|_t^{i,j,k} + (m_{Hx4}^{i,j,k}) I_{Hx}^E|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

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## Final Form of the Update Equation for $H_y$ FDTD

The update coefficients are computed before the main FDTD loop.

$$m_{Hy0}^{i,j,k} = \frac{1}{\Delta t} + \left( \frac{\sigma_x^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\epsilon_0} \right) + \frac{(\sigma_x^{H,i,j,k})(\sigma_y^{H,i,j,k})\Delta t}{4\epsilon_0^2} \quad m_{Hy1}^{i,j,k} = \frac{1}{m_{Hy0}^{i,j,k}} \left[ \frac{1}{\Delta t} - \left( \frac{\sigma_x^{H,i,j,k} + \sigma_z^{H,i,j,k}}{2\epsilon_0} \right) - \frac{(\sigma_x^{H,i,j,k})(\sigma_y^{H,i,j,k})\Delta t}{4\epsilon_0^2} \right]$$

$$m_{Hy2}^{i,j,k} = -\frac{1}{m_{Hy0}^{i,j,k}} \frac{c_0}{\mu_y^{i,j,k}} \quad m_{Hy3}^{i,j,k} = -\frac{1}{m_{Hy0}^{i,j,k}} \frac{c_0 \Delta t}{\epsilon_0} \frac{\sigma_y^{H,i,j,k}}{\mu_y^{i,j,k}} \quad m_{Hy4}^{i,j,k} = -\frac{1}{m_{Hy0}^{i,j,k}} \frac{\Delta t}{\epsilon_0^2} (\sigma_x^{H,i,j,k})(\sigma_y^{H,i,j,k})$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEy}^{i,j,k} \Big|_t = \sum_{T=0}^t C_y^E \Big|_t^{i,j,k} \quad I_{Hy}^{i,j,k} \Big|_{t-\frac{\Delta t}{2}} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_y \Big|_T^{i,j,k} \quad C_y^E \Big|_t^{i,j,k} = \frac{\tilde{E}_x \Big|_t^{i,j,k+1} - \tilde{E}_x \Big|_t^{i,j,k}}{\Delta z} - \frac{\tilde{E}_z \Big|_t^{i+1,j,k} - \tilde{E}_z \Big|_t^{i,j,k}}{\Delta x}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left( m_{Hy1} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} \right) H_y \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left( m_{Hy2} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} \right) C_y^E \Big|_t^{i,j,k} + \left( m_{Hy3} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} \right) I_{CEy} \Big|_t^{i,j,k} + \left( m_{Hy4} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} \right) I_{Hy} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

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## Final Form of the Update Equation for $H_z$ FDTD

The update coefficients are computed before the main FDTD loop.

$$m_{Hz0}^{i,j,k} = \frac{1}{\Delta t} + \left( \frac{\sigma_x^{H,i,j,k} + \sigma_y^{H,i,j,k}}{2\epsilon_0} \right) + \frac{(\sigma_x^{H,i,j,k})(\sigma_z^{H,i,j,k})\Delta t}{4\epsilon_0^2} \quad m_{Hz1}^{i,j,k} = \frac{1}{m_{Hz0}^{i,j,k}} \left[ \frac{1}{\Delta t} - \left( \frac{\sigma_x^{H,i,j,k} + \sigma_y^{H,i,j,k}}{2\epsilon_0} \right) - \frac{(\sigma_x^{H,i,j,k})(\sigma_z^{H,i,j,k})\Delta t}{4\epsilon_0^2} \right]$$

$$m_{Hz2}^{i,j,k} = -\frac{1}{m_{Hz0}^{i,j,k}} \frac{c_0}{\mu_z^{i,j,k}} \quad m_{Hz3}^{i,j,k} = -\frac{1}{m_{Hz0}^{i,j,k}} \frac{c_0 \Delta t}{\epsilon_0} \frac{\sigma_z^{H,i,j,k}}{\mu_z^{i,j,k}} \quad m_{Hz4}^{i,j,k} = -\frac{1}{m_{Hz0}^{i,j,k}} \frac{\Delta t}{\epsilon_0^2} (\sigma_x^{H,i,j,k})(\sigma_z^{H,i,j,k})$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CEz}^{i,j,k} \Big|_t = \sum_{T=0}^t C_z^E \Big|_t^{i,j,k} \quad I_{Hz}^{i,j,k} \Big|_{t-\frac{\Delta t}{2}} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} H_z \Big|_T^{i,j,k} \quad C_z^E \Big|_t^{i,j,k} = \frac{\tilde{E}_y \Big|_t^{i+1,j,k} - \tilde{E}_y \Big|_t^{i,j,k}}{\Delta x} - \frac{\tilde{E}_x \Big|_t^{i,j,k+1} - \tilde{E}_x \Big|_t^{i,j,k}}{\Delta y}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$H_z \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \left( m_{Hz1} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} \right) H_z \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left( m_{Hz2} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} \right) C_z^E \Big|_t^{i,j,k} + \left( m_{Hz3} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} \right) I_{CEz} \Big|_t^{i,j,k} + \left( m_{Hz4} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} \right) I_{Hz} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k}$$

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## Final Form of the Update Equation for $D_x$ FDTD

The update coefficients are computed before the main FDTD loop.

$$m_{Dx0}^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_y^{D,i,j,k} + \sigma_z^{D,i,j,k}}{2\epsilon_0} + \frac{(\sigma_y^{D,i,j,k})(\sigma_z^{D,i,j,k})\Delta t}{4\epsilon_0^2} \quad m_{Dx1}^{i,j,k} = \frac{1}{m_{Dx0}^{i,j,k}} \left[ \frac{1}{\Delta t} - \frac{\sigma_y^{D,i,j,k} + \sigma_z^{D,i,j,k}}{2\epsilon_0} - \frac{(\sigma_y^{D,i,j,k})(\sigma_z^{D,i,j,k})\Delta t}{4\epsilon_0^2} \right]$$

$$m_{Dx2}^{i,j,k} = -\frac{c_0}{m_{Dx0}^{i,j,k}} \quad m_{Dx3}^{i,j,k} = \frac{1}{m_{Dx0}^{i,j,k}} \frac{c_0 \Delta t \sigma_x^{D,i,j,k}}{\epsilon_0} \quad m_{Dx4}^{i,j,k} = -\frac{1}{m_{Dx0}^{i,j,k}} \frac{\Delta t}{\epsilon_0^2} (\sigma_y^{D,i,j,k})(\sigma_z^{D,i,j,k})$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHx}^{i,j,k} \Big|_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} = \sum_{T=M/2}^{t-\frac{\Delta t}{2}} C_x^H \Big|_T^{i,j,k} \quad I_{Dx}^{i,j,k} \Big|_t = \sum_{T=0}^t \tilde{D}_x \Big|_T^{i,j,k} \quad C_x^H \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_z \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z \Big|_{t+\frac{\Delta t}{2}}^{i,j-1,k}}{\Delta y} - \frac{H_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_y \Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_x \Big|_{t+\Delta t}^{i,j,k} = \left( m_{Dx1}^{i,j,k} \right) \tilde{D}_x \Big|_t^{i,j,k} + \left( m_{Dx2}^{i,j,k} \right) C_x^H \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left( m_{Dx3}^{i,j,k} \right) I_{CHx} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left( m_{Dx4}^{i,j,k} \right) I_{Dx} \Big|_{t-\Delta t}^{i,j,k}$$

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## Final Form of the Update Equation for $D_y$ FDTD

The update coefficients are computed before the main FDTD loop.

$$m_{Dy0}^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_x^{D,i,j,k} + \sigma_z^{D,i,j,k}}{2\epsilon_0} + \frac{(\sigma_x^{D,i,j,k})(\sigma_z^{D,i,j,k})\Delta t}{4\epsilon_0^2} \quad m_{Dy1}^{i,j,k} = \frac{1}{m_{Dy0}^{i,j,k}} \left[ \frac{1}{\Delta t} - \frac{\sigma_x^{D,i,j,k} + \sigma_z^{D,i,j,k}}{2\epsilon_0} - \frac{(\sigma_x^{D,i,j,k})(\sigma_z^{D,i,j,k})\Delta t}{4\epsilon_0^2} \right]$$

$$m_{Dy2}^{i,j,k} = -\frac{c_0}{m_{Dy0}^{i,j,k}} \quad m_{Dy3}^{i,j,k} = \frac{1}{m_{Dy0}^{i,j,k}} \frac{c_0 \Delta t \sigma_y^{D,i,j,k}}{\epsilon_0} \quad m_{Dy4}^{i,j,k} = -\frac{1}{m_{Dy0}^{i,j,k}} \frac{\Delta t}{\epsilon_0^2} (\sigma_x^{D,i,j,k})(\sigma_z^{D,i,j,k})$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHy} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} = \sum_{T=M/2}^{t-\frac{\Delta t}{2}} C_y^H \Big|_T^{i,j,k} \quad I_{Dy} \Big|_t^{i,j,k} = \sum_{T=0}^t \tilde{D}_y \Big|_T^{i,j,k} \quad C_y^H \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} = \frac{H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_x \Big|_{t+\frac{\Delta t}{2}}^{i,j,k-1}}{\Delta z} - \frac{H_z \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} - H_z \Big|_{t+\frac{\Delta t}{2}}^{i-1,j,k}}{\Delta x}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_y \Big|_{t+\Delta t}^{i,j,k} = \left( m_{Dy1}^{i,j,k} \right) \tilde{D}_y \Big|_t^{i,j,k} + \left( m_{Dy2}^{i,j,k} \right) C_y^H \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + \left( m_{Dy3}^{i,j,k} \right) I_{CHy} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + \left( m_{Dy4}^{i,j,k} \right) I_{Dy} \Big|_{t-\Delta t}^{i,j,k}$$

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## Final Form of the Update Equation for $D_z$ FDTD

The update coefficients are computed before the main FDTD loop.

$$m_{Dz0}^{i,j,k} = \frac{1}{\Delta t} + \frac{\sigma_x^D |^{i,j,k} + \sigma_y^D |^{i,j,k}}{2\epsilon_0} + \frac{(\sigma_x^D |^{i,j,k})(\sigma_y^D |^{i,j,k})\Delta t}{4\epsilon_0^2} \quad m_{Dz1}^{i,j,k} = \frac{1}{m_{Dz0}^{i,j,k}} \left[ \frac{1}{\Delta t} - \frac{\sigma_x^D |^{i,j,k} + \sigma_y^D |^{i,j,k}}{2\epsilon_0} - \frac{(\sigma_x^D |^{i,j,k})(\sigma_y^D |^{i,j,k})\Delta t}{4\epsilon_0^2} \right]$$

$$m_{Dz2}^{i,j,k} = \frac{c_0}{m_{Dz0}^{i,j,k}} \quad m_{Dz3}^{i,j,k} = \frac{1}{m_{Dz0}^{i,j,k}} \frac{c_0 \Delta t \sigma_z^D |^{i,j,k}}{\epsilon_0} \quad m_{Dz4}^{i,j,k} = -\frac{1}{m_{Dz0}^{i,j,k}} \frac{\Delta t}{\epsilon_0^2} (\sigma_x^D |^{i,j,k})(\sigma_y^D |^{i,j,k})$$

The integration terms are computed inside the main FDTD loop, but before the update equation.

$$I_{CHz}^{i,j,k} \Big|_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} = \sum_{T=\Delta t/2}^{t-\frac{\Delta t}{2}} C_z^H \Big|_T \quad I_{Dz}^{i,j,k} = \sum_{T=0}^t \tilde{D}_z \Big|_T \quad C_z^H \Big|_{t+\frac{\Delta t}{2}} = \frac{H_y \Big|_{t+\frac{\Delta t}{2}} - H_y \Big|_{t-\frac{\Delta t}{2}}}{\Delta x} - \frac{H_x \Big|_{t+\frac{\Delta t}{2}} - H_x \Big|_{t-\frac{\Delta t}{2}}}{\Delta y}$$

The update equation is computed inside the main FDTD loop immediately after the integration terms are updated.

$$\tilde{D}_z \Big|_{t+\Delta t}^{i,j,k} = (m_{Dz1}^{i,j,k}) \tilde{D}_z \Big|_t^{i,j,k} + (m_{Dz2}^{i,j,k}) C_z^H \Big|_{t+\frac{\Delta t}{2}}^{i,j,k} + (m_{Dz3}^{i,j,k}) I_{CHz} \Big|_{t-\frac{\Delta t}{2}}^{i,j,k} + (m_{Dz4}^{i,j,k}) I_{Dz} \Big|_{t-\Delta t}^{i,j,k}$$

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## Final Update Equations for $E_x$ , $E_y$ , and $E_z$ FDTD

The update coefficients are computed before the main FDTD loop.

$$m_{Ex1}^{i,j,k} = \frac{1}{\mathcal{E}_{xx} |^{i,j,k}} \quad m_{Ey1}^{i,j,k} = \frac{1}{\mathcal{E}_{yy} |^{i,j,k}} \quad m_{Ez1}^{i,j,k} = \frac{1}{\mathcal{E}_{zz} |^{i,j,k}}$$

The update equations are computed inside the main FDTD loop.

$$\tilde{E}_x \Big|_{t+\Delta t}^{i,j,k} = (m_{Ex1}^{i,j,k}) \tilde{D}_x \Big|_{t+\Delta t}^{i,j,k}$$

$$\tilde{E}_y \Big|_{t+\Delta t}^{i,j,k} = (m_{Ey1}^{i,j,k}) \tilde{D}_y \Big|_{t+\Delta t}^{i,j,k}$$

$$\tilde{E}_z \Big|_{t+\Delta t}^{i,j,k} = (m_{Ez1}^{i,j,k}) \tilde{D}_z \Big|_{t+\Delta t}^{i,j,k}$$

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