EE 5303
Electromagnetic Analysis Using Finite-Difference Time-Domain

Lecture #24

Scattering Analysis

Lecture Outline

• Introduction
• FDTD for Scattering Simulations
• Formulation of NF2FF Transformation
• Implementation
• NF2FF in Two Dimensions
• Simulation Examples
Introduction

Animation of Scattering Analysis

- Finite object
- TF/SF on all surrounding sides
- Interested primarily in the scattered-field
- Usually the goal is to quantify scattering in the far field.
What is NF2FF Transformation? (1 of 2)

Suppose we want to know the electromagnetic fields VERY far away from an object.

How do we do this? Do we just use an extremely large grid?

NO!!

What is NF2FF Transformation? (2 of 2)

Instead, we perform a small simulation and use the NF2FF technique to calculate the fields very far away. This is MUCH more efficient!
Representing the Far Field

Usually the far field is plotted in cylindrical (2D) or spherical (3D) coordinates.

Basic Outline of Scattering Analysis

Step 1 – Build object on grid
Step 2 – Simulate object
Step 3 – Calculate steady-state field
Step 4 – Perform NF2FF
Step 5 – Post process
Our source is bounded on all sides so that the entire outer portion of the grid is scattered-field.
Recall the Curl Calculations

**E Mode**
\[
C^E_{i,j} = \frac{\tilde{E}_i^{j+1} - \tilde{E}_i^{j-1}}{\Delta y}
\]
\[
C^E_{i,j} = \frac{\tilde{E}_i^{j+1} - \tilde{E}_i^{j-1}}{\Delta x}
\]
\[
C^E_{z,k} = \frac{\tilde{H}_z^{j+1} - \tilde{H}_z^{j-1}}{\Delta x} - \frac{\tilde{H}_i^{j+1} - \tilde{H}_i^{j-1}}{\Delta y}
\]

**H Mode**
\[
C^H_{i,j} = \frac{\tilde{E}_i^{j+1} - \tilde{E}_i^{j-1}}{\Delta y}
\]
\[
C^H_{i,j} = \frac{\tilde{E}_i^{j+1} - \tilde{E}_i^{j-1}}{\Delta x}
\]
\[
C^H_{z,k} = \frac{\tilde{H}_z^{j+1} - \tilde{H}_z^{j-1}}{\Delta x} - \frac{\tilde{H}_i^{j+1} - \tilde{H}_i^{j-1}}{\Delta y}
\]

Note: the terms in red identify terms that require modification in the TF/SF framework.

---

**x-lo Side**

On the x-lo side, only $CE_y$ and $CH_z$ require modification.

\[
C^E_{x,\rightarrow} = \frac{\tilde{E}_x^{\rightarrow, j,k}}{\Delta x}
\]
\[
C^E_{x,\rightarrow} = \frac{\tilde{E}_x^{\rightarrow, j,k}}{\Delta x}
\]
\[
C^H_{z,\rightarrow} = \frac{\tilde{H}_z^{\rightarrow, j,k}}{\Delta x}
\]
\[
C^H_{z,\rightarrow} = \frac{\tilde{H}_z^{\rightarrow, j,k}}{\Delta x}
\]

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**x-hi Side**

On the x-hi side, only $CE_y$ and $CH_z$ require modification.

$$C^E_y \bigg|^{l_{2},j}_{l_{1},j} = -\frac{\Delta x}{\Delta} \left( \tilde{E}_z \bigg|^{l_{2}+1,1}_{l_{1}+1,1} - \tilde{E}_z \bigg|^{l_{2},1}_{l_{1},1} \right)$$

$$= \left\{ -\frac{\Delta x}{\Delta} \tilde{E}_z \bigg|^{l_{2}+1,1}_{l_{1}+1,1} - \tilde{E}_z \bigg|^{l_{2},1}_{l_{1},1} \right\} - \frac{\tilde{E}_z,src \bigg|^{l_{2}+1,1}_{l_{1}+1,1}}{\Delta x}$$

$$C^H_z \bigg|^{l_{2}+1,j}_{l_{1}+1,j} = \frac{\Delta x}{\Delta y} \left( H_y \bigg|^{l_{2}+1,1}_{l_{1}+1,1} - H_y \bigg|^{l_{2},1}_{l_{1},1} \right)$$

$$= \left\{ \frac{H_y \bigg|^{l_{2}+1,1}_{l_{1}+1,1} - H_y \bigg|^{l_{2},1}_{l_{1},1}}{\Delta x} - \frac{H_y \bigg|^{l_{2}+1,1}_{l_{1}+1,1} - H_y \bigg|^{l_{2},1}_{l_{1},1}}{\Delta y} \right\} + \frac{H_y,src \bigg|^{l_{2}+1,1}_{l_{1}+1,1}}{\Delta x}$$

**y-lo Side**

On the y-lo side, only $CE_x$ and $CH_z$ require modification.

$$C^E_x \bigg|^{l_{1},j}_{l_{1},j} = \frac{\Delta y}{\Delta} \left( \tilde{E}_z \bigg|^{l_{1},j}_{l_{1},j} - \tilde{E}_z \bigg|^{l_{1},j-1}_{l_{1},j-1} \right)$$

$$= \left\{ \frac{\Delta x}{\Delta} \tilde{E}_z \bigg|^{l_{1},j}_{l_{1},j} - \tilde{E}_z \bigg|^{l_{1},j-1}_{l_{1},j-1} \right\} - \frac{\tilde{E}_z,src \bigg|^{l_{1},j}_{l_{1},j}}{\Delta y}$$

$$C^H_z \bigg|^{l_{1},j}_{l_{1},j} = \frac{\Delta x}{\Delta y} \left( H_x \bigg|^{l_{1},j}_{l_{1},j} - H_x \bigg|^{l_{1},j-1}_{l_{1},j-1} \right)$$

$$= \left\{ \frac{H_x \bigg|^{l_{1},j}_{l_{1},j} - H_x \bigg|^{l_{1},j-1}_{l_{1},j-1}}{\Delta x} - \frac{H_x \bigg|^{l_{1},j}_{l_{1},j} - H_x \bigg|^{l_{1},j-1}_{l_{1},j-1}}{\Delta y} \right\} + \frac{H_x,src \bigg|^{l_{1},j}_{l_{1},j}}{\Delta y}$$
y-hi Side

On the y-hi side, only $CEx$ and $CHz$ require modification.

$$C_x^{i,j_2} = \left. \frac{\vec{E}_z^{i,j_2+1} - \vec{E}_z^{i,j_2}}{\Delta y} \right|_{y}$$

$$= \frac{\vec{E}_z^{i,j_2+1} - \vec{E}_z^{i,j_2}}{\Delta y} + \frac{\vec{E}_{z,arc}^{i,j_2+1}}{\Delta y}$$

$$C_x^{i,j_2+1} = \left. \frac{H_y^{i,j_2+1} - H_y^{i-1,j_2+1}}{\Delta x} \right|_{x} - \frac{H_x^{i,j_2+1} - H_x^{i,j_2}}{\Delta y}$$

$$= \frac{H_y^{i,j_2+1} - H_y^{i-1,j_2+1}}{\Delta x} - \frac{H_x^{i,j_2+1} - H_x^{i,j_2}}{\Delta y}$$

Deriving a Sinusoidal Source Function

The general expression for the normalized electric field component of a plane wave is

$$\vec{E}(x, y; t) = \vec{P} \cos \left( \omega t - \vec{k} \cdot \vec{r} \right) = \vec{P} \cos \left( \omega t - k_x x - k_y y \right)$$

The equivalent expression can be derived for the magnetic field by substituting the above expression into Faraday’s law.

$$\vec{H} = \sqrt{\frac{\epsilon_r}{\mu_r}} \left( \vec{k} \times \vec{P} \right) \cos \left( \omega t - \vec{k} \cdot \vec{r} \right) = \sqrt{\frac{\epsilon_r}{\mu_r}} \left( \vec{k} \times \vec{P} \right) \cos \left( \omega t - k_x x - k_y y \right)$$
Sinusoidal Source Functions for the E Mode

The E mode has $P_x = P_y = 0$ and we set $P_z = 1$. Thus, the source functions reduce to

$$\tilde{E}_z(x, y, t) = \cos(\omega t - k_x x - k_y y)$$

$$H_x(x, y, t) = \hat{k}_y \sqrt{\frac{\varepsilon_r}{\mu_r}} \cos(\omega t - k_x x - k_y y)$$

$$H_y(x, y, t) = -\hat{k}_x \sqrt{\frac{\varepsilon_r}{\mu_r}} \cos(\omega t - k_x x - k_y y)$$

Numerical Equations for Sinusoidal Source Functions for the E Mode

In terms of array indices, the source functions are

$$\tilde{E}_{x,\text{src}}^{i,j} = \cos(\omega T \Delta t - k_x i \Delta x - k_y j \Delta y)$$

$$H_{x,\text{src}}^{i,j} = \hat{k}_y \sqrt{\frac{\varepsilon_r}{\mu_r}} \cos(\omega (T + 0.5) \Delta t - k_x (i - 0.5) \Delta x - k_y j \Delta y)$$

$$\tilde{E}_{y,\text{src}}^{i,j} = \hat{k}_x \sqrt{\frac{\varepsilon_r}{\mu_r}} \cos(\omega T \Delta t - k_x i \Delta x - k_y j \Delta y)$$

$$H_{y,\text{src}}^{i,j} = \hat{k}_x \sqrt{\frac{\varepsilon_r}{\mu_r}} \cos(\omega (T + 0.5) \Delta t - k_x (i + 0.5) \Delta x - k_y j \Delta y)$$

$$\tilde{E}_{z,\text{src}}^{i,j} = \cos(\omega T \Delta t - k_x i \Delta x - k_y j \Delta y)$$

$$H_{z,\text{src}}^{i,j} = \hat{k}_y \sqrt{\frac{\varepsilon_r}{\mu_r}} \cos(\omega (T + 0.5) \Delta t - k_x (j - 0.5) \Delta y)$$
Code Snippets for Sinusoidal Source

% TF/SF FOR CURL OF E
Ez0 = RAMP(T);
% xlo
Ezsrc = Ez0*cos(2*pi*f0*T*dt - kx*nxlo*dx - ky*[nylo:nyhi]*dy);
CEy(nxlo-1,nylo:nyhi) = CEy(nxlo-1,nylo:nyhi) + Ezsrc/dx;
% xhi
Ezsrc = Ez0*cos(2*pi*f0*T*dt - kx*(nxhi + 1)*dx - ky*[nylo:nyhi]*dy);
CEy(nxhi,nylo:nyhi) = CEy(nxhi,nylo:nyhi) - Ezsrc/dx;
% ylo
Ezsrc = Ez0*cos(2*pi*f0*T*dt - kx*[nxlo:nxhi]'*dx - ky*nylo*dy);
CEx(nxlo:nxhi,nylo-1) = CEx(nxlo:nxhi,nylo-1) - Ezsrc/dy;
% yhi
Ezsrc = Ez0*cos(2*pi*f0*T*dt - kx*[nxlo:nxhi]'*dx - ky*(nyhi + 1)*dy);
CEx(nxlo:nxhi,nyhi) = CEx(nxlo:nxhi,nyhi) + Ezsrc/dy;

% TF/SF FOR CURL OF H
Hx0 = +(ky/k0)*RAMP(T);
Hy0 = -(kx/k0)*RAMP(T);
% xlo
Hysrc = Hy0*cos(2*pi*f0*[T + 0.5]*dt - kx*[nxlo - 0.5]*dx - ky*[nylo:nyhi]*dy);
CHz(nxlo,nylo:nyhi) = CHz(nxlo,nylo:nyhi) - Hysrc/dx;
% xhi
Hysrc = Hy0*cos(2*pi*f0*[T + 0.5]*dt - kx*[nxhi + 0.5]*dx - ky*[nylo:nyhi]*dy);
CHz(nxhi+1,nylo:nyhi) = CHz(nxhi+1,nylo:nyhi) + Hysrc/dx;
% ylo
Hxsrc = Hx0*cos(2*pi*f0*[T + 0.5]*dt - kx*[nxlo:nxhi]'*dx - ky*[nylo - 0.5]*dy);
CHz(nxlo:nxhi,nylo) = CHz(nxlo:nxhi,nylo) + Hxsrc/dy;
% yhi
Hxsrc = Hx0*cos(2*pi*f0*[T + 0.5]*dt - kx*[nxlo:nxhi]'*dx - ky*[nyhi + 0.5]*dy);
CHz(nxlo:nxhi,nyhi+1) = CHz(nxlo:nxhi,nyhi+1) - Hxsrc/dy;

Animation of CW Source

STEP 2 of 500
Calculating $k_x$ and $k_y$ (1 of 2)

The TF/SF spans a large enough amount of space that the numerical dispersion due to the Yee grid causes problems. You must compensate for this.

![Uncompensated and Compensated plots]

Calculating $k_x$ and $k_y$ (2 of 2)

Step 1 – Define $\theta$ in dashboard.

Step 2 – Calculate the refractive index where source is injected.

\[ n_{src} = \sqrt{\mu_{r,src} \varepsilon_{r,src}} \]

Step 3 – Calculate $k_x$ and $k_y$.

\[ k_x = k_0 n_{src} \cos \theta \quad k_y = k_0 n_{src} \sin \theta \]

Step 3 – Calculate “fudge factor” $f$.

\[ f = \frac{c_0 \Delta t}{n_{src} \sin (\pi f_0 \Delta t)} \sqrt{\left[ (1/\Delta x) \sin (k_x \Delta x/2) \right]^2 + \left[ (1/\Delta y) \sin (k_y \Delta y/2) \right]^2} \]

Step 4 – Adjust $k_x$ and $k_y$.

\[ \tilde{k}_x = f k_x \quad \tilde{k}_y = f k_y \]

Note: An alternate method is described for the Gaussian source that usually works better.
Analytical Equations for a Gaussian Pulse

For a Gaussian pulse, the field components are

\[ \vec{E}_z (x, y; t) = \exp \left[ -\left( \frac{t - \delta t}{\tau} \right)^2 \right] \]

\[ H_x (x, y; t) = \hat{k}_y \exp \left[ -\left( \frac{t - \delta t}{\tau} \right)^2 \right] \]

\[ H_y (x, y; t) = -\hat{k}_x \exp \left[ -\left( \frac{t - \delta t}{\tau} \right)^2 \right] \]

\[ \delta t = \left( \hat{k}_x x + \hat{k}_y y \right) / c_0 \]

This time delay ensures the pulse begins just outside of the TF/SF region regardless of the angle of incidence.

Numerical Equations for a Gaussian Pulse

For a Gaussian pulse, the field components are

\[ \vec{E}_z (i; j; T) = \exp \left[ -\frac{(T - \Delta t - \left[ \hat{k}_x i \Delta x + \hat{k}_y j \Delta y \right] / c_0)^2}{\tau} \right] \]

\[ H_x (i; j; T) = \hat{k}_y \exp \left[ -\frac{(T + 0.5) \Delta t - \left[ \hat{k}_x i \Delta x + \hat{k}_y (j + 0.5) \Delta y \right] / c_0)^2}{\tau} \right] \]

\[ H_y (i; j; T) = -\hat{k}_x \exp \left[ -\frac{(T + 0.5) \Delta t - \left[ \hat{k}_x (i + 0.5) \Delta x + \hat{k}_y j \Delta y \right] / c_0)^2}{\tau} \right] \]
Code Snippets for Gaussian Source

% TF/SF FOR CURL OF E
Ez0 = exp(-((T*dt - T0)/tau).^2);
Ez0 = Ez0(1:2:Nx2,1:2:Ny2);
xlo
Ezsrc = Ez0(nxlo,nylo:nyhi);

%xhi
Ezsrc = Ez0(nxhi+1,nylo:nyhi);

CEy(nxlo-1,nylo:nyhi) = CEy(nxlo-1,nylo:nyhi) + Ezsrc/dx;

%ylo
Ezsrc = Ez0(nxlo:nxhi,nylo);

%yhi
Ezsrc = Ez0(nxlo:nxhi,nyhi+1);

CEx(nxlo:nxhi,nylo-1) = CEx(nxlo:nxhi,nylo-1) - Ezsrc/dy;

% TF/SF
H0 = exp(-((T + 0.5)*dt - T0)/tau).^2;
Hx0 = + (ky/k0)*H0(2:2:Nx2,1:2:Ny2);
ylo
Hysrc = Hx0(nxlo-1,nylo:nyhi);

%yhi
Hysrc = Hx0(nxlo:nxhi,nyhi);

CHz(nxlo,nylo:nyhi) = CHz(nxlo,nylo:nyhi) - Hysrc/dx;

% CALCULATE TIME GRADIENT ACROSS 2x2 GRID
delx = Sx*cos(theta);dely = Sy*sin(theta);
T0 = (cos(theta)*(X2 + delx) + sin(theta)*(Y2 + dely))/c0;

Animation of Gaussian Pulse Source
**Compensating for Dispersion for a Gaussian Source (1 of 2)**

The TF/SF spans a large enough amount of space that the numerical dispersion due to the Yee grid causes problems. You must compensate for this.
Compensating for Dispersion for a Gaussian Source (2 of 2)

Step 1 – Define $\theta$ in dashboard.

Step 2 – Calculate the refractive index where source is injected.

$$n_{src} = \sqrt{\mu_{r,src} \varepsilon_{r,src}}$$

Step 3 – Calculate $k_x$ and $k_y$.

$$k_x = k_0 n_{src} \cos \theta$$
$$k_y = k_0 n_{src} \sin \theta$$

Step 4 – Calculate “fudge factor” $f$.

$$f = \frac{c_0 \Delta t}{n_{src} \sin(\pi f_0 \Delta t)} \sqrt{\left(\frac{1}{\Delta x \sin(k_x \Delta x / 2)}\right)^2 + \left(\frac{1}{\Delta y \sin(k_y \Delta y / 2)}\right)^2}$$

Step 5 – Adjust material arrays $ER2$ and $UR2$ according to $f$.

$$UR2 = f \times UR2;$$
$$ER2 = f \times ER2;$$

Note: This method of compensate works for the sinusoidal source as well.

Formulation of NF2FF Transformation
### Electromagnetic Boundary Conditions

#### Magnetic Field at an Interface

\[ \vec{H}_{2,T} = \vec{H}_{1,T} + \vec{J}_s \]

\[ \vec{H}_{2,T} - \vec{H}_{1,T} = \vec{J}_s \]

\[ \hat{n} \times \vec{H}_2 - \hat{n} \times \vec{H}_1 = \vec{J}_s \]

\[ \hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{J}_s \]

\[ \vec{J}_s = \hat{n} \times (\vec{H}_2 - \vec{H}_1) \]

\[ \hat{n} \equiv \text{surface normal pointing from 1 to 2} \]

#### Electric Field at an Interface

\[ \vec{E}_{2,T} = \vec{E}_{1,T} - \vec{M}_s \]

\[ \vec{E}_{2,T} - \vec{E}_{1,T} = -\vec{M}_s \]

\[ \hat{n} \times \vec{E}_2 - \hat{n} \times \vec{E}_1 = -\vec{M}_s \]

\[ \hat{n} \times (\vec{E}_2 - \vec{E}_1) = -\vec{M}_s \]

\[ \vec{M}_s = -\hat{n} \times (\vec{E}_2 - \vec{E}_1) \]

### Surface Equivalence Theorem

The far field \( E \) and \( H \) is completely described by the surface currents \( \vec{J}_s \) and \( \vec{M}_s \) flowing around a closed surface \( S \).

\[ \vec{E}, \vec{H} \]

[Diagram showing surface currents and fields]

\[ \vec{E}, \vec{H} \]

\[ \vec{E}, \vec{H} \]

\[ \vec{E}, \vec{H} \]
Calculating the Surface Currents

Since we are only interested in the far fields $E$ and $H$, we are free to let $E_1$ and $H_1$ be whatever is convenient.

Let $\vec{E}_1 = \vec{H}_1 = 0$.

We can now calculate the surface currents as

$$\vec{J}_s = \hat{n} \times \vec{H}$$
$$\vec{M}_s = -\hat{n} \times \vec{E}$$

Note: Calculating $J_s$ and $M_s$ in this way makes it only possible to calculate fields outside of the surface.

Vector Potentials

We wish to calculate the fields very far from the surface $S$. It is useful to do this using the vector potentials.

$$\vec{A} = \frac{\mu}{4\pi} \int_S \vec{J}_s \frac{e^{ikR}}{R} \, ds'$$
$$\vec{F} = \frac{\varepsilon}{4\pi} \int_S \vec{M}_s \frac{e^{ikR}}{R} \, ds'$$

where $R \equiv$ distance from point on surface $s'$ to the observation point.
Far Field Approximation

In the far field, \( R \) is extremely large.

\[
R = |\vec{r} - \vec{r}'| \approx \begin{cases} 
    r - r' \cos \psi & \text{for phase calculations} \\
    r & \text{for amplitude calculations}
\end{cases}
\]

Note:
\[
r' \cos \psi = \vec{r}' \cdot \hat{r} \quad \vec{r} = \frac{\vec{r}'}{|\vec{r}'|}
\]

Vector Potentials in the Far Field

Using the far field approximation for \( R \), the equations for calculating the vector potentials become

\[
\vec{A} \approx \frac{\mu}{4\pi} \int_S \vec{J} e^{jk(r-r' \cos \psi)} dr' \quad \vec{F} \approx \frac{\varepsilon}{4\pi} \int_S \vec{M} e^{jk(r-r' \cos \psi)} dr'
\]

\[
\approx \frac{\mu e^{ikr}}{4\pi r} \int_S \vec{J} e^{-ikr' \cos \psi} dr' \quad \approx \frac{\varepsilon e^{ikr}}{4\pi r} \int_S \vec{M} e^{-ikr' \cos \psi} dr'
\]

Last, we write these equations in spherical coordinates.

\[
\vec{A}(r, \theta, \phi) \approx \frac{\mu e^{ikr}}{4\pi r} \vec{N}(\theta, \phi) \quad \vec{F}(r, \theta, \phi) \approx \frac{\varepsilon e^{ikr}}{4\pi r} \vec{L}(\theta, \phi)
\]

\[
\vec{N}(\theta, \phi) \approx \int_S \vec{J} e^{-ikr' \cos \psi} dr' \quad \vec{L}(\theta, \phi) \approx \int_S \vec{M} e^{-ikr' \cos \psi} dr'
\]

It is important to note the \( r, \phi, \) and \( \theta \) dependencies in these equations.
**Cartesian to Spherical Coordinates (1 of 2)**

Standard FDTD produces field quantities in Cartesian coordinates. This means the current terms $J_x$ and $M_x$ will be in Cartesian coordinates. Usually far-field analysis is done in spherical coordinates so these functions must be converted to spherical coordinates.

\[
\begin{bmatrix}
J_r \\
J_\theta \\
J_\phi
\end{bmatrix}
= \begin{bmatrix}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{bmatrix}
\begin{bmatrix}
J_x \\
J_y \\
J_z
\end{bmatrix}
\]

\[
\vec{N} = N_r \hat{r} + N_\theta \hat{\theta} + N_\phi \hat{\phi}
\]

\[
N_r = \iint_S (J_x \sin \theta \cos \phi + J_y \sin \theta \sin \phi + J_z \cos \theta) e^{-jkr' \cos \phi} dS'
\]

\[
N_\theta = \iint_S (J_x \cos \theta \cos \phi + J_y \cos \theta \sin \phi - J_z \sin \theta) e^{-jkr' \cos \phi} dS'
\]

\[
N_\phi = \iint_S (-J_x \sin \phi + J_y \cos \phi) e^{-jkr' \cos \phi} dS'
\]

**Cartesian to Spherical Coordinates (2 of 2)**

And the calculation for $\vec{L}(\theta, \phi)$ is

\[
\begin{bmatrix}
M_r \\
M_\theta \\
M_\phi
\end{bmatrix}
= \begin{bmatrix}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

\[
\vec{L} = L_r \hat{r} + L_\theta \hat{\theta} + L_\phi \hat{\phi}
\]

\[
L_r = \iint_S (M_x \sin \theta \cos \phi + M_y \sin \theta \sin \phi + M_z \cos \theta) e^{-jkr' \cos \phi} dS'
\]

\[
L_\theta = \iint_S (M_x \cos \theta \cos \phi + M_y \cos \theta \sin \phi - M_z \sin \theta) e^{-jkr' \cos \phi} dS'
\]

\[
L_\phi = \iint_S (-M_x \sin \phi + M_y \cos \phi) e^{-jkr' \cos \phi} dS'
\]
\( \tilde{N}(\theta, \phi) \) and \( \tilde{L}(\theta, \phi) \) are far field quantities so they are independent of the choice of the surface \( S \).

Let's pick this one for now.

The current terms \( J_s \) and \( M_s \) are surface currents and always tangential to the surface.

So when we choose \( S \) to be a sphere, the surface currents have no radial components.

\[
J_r = M_r = 0 \quad \text{when } S \text{ is a sphere}
\]
We see from the following equations that $\tilde{N}$ and $\tilde{L}$ take one the same vector components as $\tilde{J}_x$ and $\tilde{M}_x$.

$$\tilde{N}(\theta, \phi) \equiv \iint_S \tilde{J}_x e^{-jkr\cos \psi} \, ds'$$
$$\tilde{L}(\theta, \phi) \equiv \iint_S \tilde{M}_x e^{-jkr\cos \psi} \, ds'$$

Therefore, $\tilde{N}$ and $\tilde{L}$ have no radial components.

We also recognize that this will always be the case since the far-field is fixed and independent of the choice of $S$.

$$N_r = L_r = 0 \quad \text{always}$$

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**Final Expressions for $N$ and $L$**

$$N_r(\theta, \phi) = 0$$
$$N_{\theta}(\theta, \phi) = \iint_S (J_x \cos \theta \cos \phi + J_y \cos \theta \sin \phi - J_z \sin \theta) e^{-jkr\cos \psi} \, ds'$$
$$N_{\phi}(\theta, \phi) = \iint_S (-J_x \sin \phi + J_y \cos \phi) e^{-jkr\cos \psi} \, ds'$$

$$L_r(\theta, \phi) = 0$$
$$L_{\theta}(\theta, \phi) = \iint_S (M_x \cos \theta \cos \phi + M_y \cos \theta \sin \phi - M_z \sin \theta) e^{-jkr\cos \psi} \, ds'$$
$$L_{\phi}(\theta, \phi) = \iint_S (-M_x \sin \phi + M_y \cos \phi) e^{-jkr\cos \psi} \, ds'$$
Calculating the Far Fields (1 of 3)

The electric and magnetic fields, \( E \) and \( H \), are calculated from the vector potentials according to

\[
\vec{H} = \frac{1}{\mu} \nabla \times \vec{A} + j \omega \vec{F} + \frac{j}{\omega \mu \varepsilon} \nabla (\nabla \cdot \vec{E}) \\
\vec{E} = -\frac{1}{\varepsilon} \nabla \times \vec{F} + j \omega \vec{A}
\]

The \( \nabla (\nabla \cdot \vec{E}) \) terms produce equations with \( 1/r^2 \) dependence. In the far field \( r \) is very large so these terms vanish.

\[
\vec{H} \approx \frac{1}{\mu} \nabla \times \vec{A} + j \omega \vec{F} \\
\vec{E} \approx -\frac{1}{\varepsilon} \nabla \times \vec{F} + j \omega \vec{A}
\]

Calculating the Far Fields (2 of 3)

The curl terms become

\[
\nabla \times \vec{A} = \nabla \times \left( \frac{\mu e^{jkr}}{4\pi r} \hat{N} \right) \\
= \frac{\mu}{4\pi} \nabla \times \left( \frac{e^{jkr}}{r} \hat{N} \right) \\
= \frac{\mu}{4\pi} jke^{jkr} (N_\theta \hat{\theta}_\phi - N_\phi \hat{\phi}_\theta) \\
= jk (A_\theta \hat{\theta}_\phi - A_\phi \hat{\phi}_\theta)
\]

\[
\nabla \times \vec{F} = \nabla \times \left( \frac{\varepsilon e^{jkr}}{4\pi r} \hat{N} \right) \\
= \frac{\varepsilon}{4\pi} \nabla \times \left( \frac{e^{jkr}}{r} \hat{N} \right) \\
= \frac{\varepsilon}{4\pi} jke^{jkr} \left( L_\theta \hat{\theta}_\phi - L_\phi \hat{\phi}_\theta \right) \\
= jk (F_\theta \hat{\theta}_\phi - F_\phi \hat{\phi}_\theta)
\]

The electric and magnetic fields are now

\[
\vec{H} = \frac{jk}{\mu} \left( A_\theta \hat{\theta}_\phi - A_\phi \hat{\phi}_\theta \right) + j \omega \vec{F} \\
\vec{E} = -\frac{jk}{\varepsilon} \left( F_\theta \hat{\theta}_\phi - F_\phi \hat{\phi}_\theta \right) + j \omega \vec{A}
\]
Calculating the Far Fields (3 of 3)

\[ \tilde{H} = \frac{jk}{\mu} (A_\phi \hat{a}_\phi - A_\theta \hat{a}_\theta) + j\omega \tilde{F} \]
\[ \tilde{E} = \frac{jk}{\varepsilon} (F_\phi \hat{a}_\phi - F_\theta \hat{a}_\theta) + j\omega \tilde{A} \]

We now combine vector components.

\[ \tilde{H} = -j\omega \left( \frac{1}{\eta} A_\phi - F_\phi \right) \hat{a}_\phi + j\omega \left( \frac{1}{\eta} A_\theta + F_\theta \right) \hat{a}_\theta \]
\[ \tilde{E} = j\omega \left( A_\phi + \eta F_\phi \right) \hat{a}_\phi + j\omega \left( A_\theta - \eta F_\theta \right) \hat{a}_\theta \]

And last we put these equations in terms of \( N \) and \( L \).

- \( H_r = 0 \)
- \( H_\theta = -\frac{jke^{jkr}}{4\pi r} \left( N_\phi - \frac{1}{\eta} L_\phi \right) \)
- \( H_\phi = \frac{jke^{jkr}}{4\pi r} \left( N_\theta + \frac{1}{\eta} L_\theta \right) \)
- \( E_r = 0 \)
- \( E_\theta = \frac{jke^{jkr}}{4\pi r} \left( \eta N_\phi + L_\phi \right) \)
- \( E_\phi = \frac{jke^{jkr}}{4\pi r} \left( \eta N_\theta - L_\theta \right) \)

Impedance of the far-field medium:

\[ \eta = \sqrt{\frac{\mu}{\varepsilon}} \]

Radar Cross Section

The power in the radiated field can be calculated from the far field quantities as

\[ P_{\text{rad}} = \frac{1}{2} \text{Re} \left[ E_\theta H_\theta^* \right] + \frac{1}{2} \text{Re} \left[ -E_\phi H_\phi^* \right] \]
\[ = \frac{1}{2\eta} \left( \frac{k}{4\pi r} \right)^2 \left( |\eta N_\phi + L_\phi|^2 + |\eta N_\theta - L_\theta|^2 \right) \]

The radar cross section (RCS) in three dimensions is

\[ \text{RCS}(\theta, \phi) = \lim_{r \to \infty} \left( 4\pi r^2 \frac{P_{\text{rad}}}{P_{\text{inc}}} \right) \]
\[ = \frac{k^2}{8\pi\eta P_{\text{inc}}} \left( |\eta N_\phi + L_\phi|^2 + |\eta N_\theta - L_\theta|^2 \right) \]
Flow of Equations for 3D NF2FF

Step 1 – Calculate current terms around a closed surface.

\[ \vec{J}_s = \hat{n} \times \vec{H} \quad \vec{M}_s = -\hat{n} \times \vec{E} \]

Step 2 – Loop over all values of \( \theta \) and \( \phi \)

a) Calculate \( N \) and \( L \) in the far field

\[ N_s = \int (J, \cos \theta \cos \phi - J, \sin \theta \sin \phi) e^{-jkr} \, d\sigma \]
\[ L_s = \int (M, \cos \theta \cos \phi + M, \sin \theta \sin \phi) e^{-jkr} \, d\sigma \]

b) Calculate \( E \) and \( H \) in the far field

\[ H_\phi = 0 \]
\[ H_\theta = -j\mu_0 \frac{1}{4\pi} \left( N_\theta - \frac{1}{9} N_\phi \right) \]
\[ H_\rho = j\mu_0 \frac{1}{4\pi} \left( N_\rho + \frac{1}{9} N_\phi \right) \]
\[ E_\phi = 0 \]
\[ E_\theta = j\varepsilon_0 \frac{1}{4\pi} (\eta N_\phi + L_\theta) \]
\[ E_\rho = j\varepsilon_0 \frac{1}{4\pi} (\eta N_\phi - L_\theta) \]

c) Calculate RCS or whatever else you need

\[ \text{RCS}(\theta, \phi) = \frac{k^2}{8\pi\mu_0} \left( |N_\theta + L_\theta|^2 + |N_\phi - L_\phi|^2 \right) \]

NF2FF in Two Dimensions
Surface Currents (1 of 2)

We restrict our analysis to the x-y plane.

This means the normal vector is restricted to the x-y plane.

\[ \hat{n} = n_x \hat{a}_x + n_y \hat{a}_y \]

This also means that the spherical coordinate parameter \( \theta \) is zero.

\[ \theta = 0 \]

Therefore, we write \( N \) and \( L \) only as functions of \( \phi \).

\[ N_\phi (\phi), \ N_\phi (\phi), \ L_\phi (\phi), \ L_\phi (\phi) \]

Surface Integrals to Line Integrals (1 of 2)

For the general case, we integrate the current terms \( J_s \) and \( M_s \) over a closed surface to obtain \( N(\phi) \) and \( L(\phi) \) in the far field.

\[ N_\phi (\phi) = \iiint_S (\cdots) ds' \quad \quad \quad L_\phi (\phi) = \iiint_S (\cdots) ds' \]

\[ N_\phi (\phi) = \iiint_S (\cdots) ds' \quad \quad \quad L_\phi (\phi) = \iiint_S (\cdots) ds' \]

In FDTD, it is most convenient to setup our integrating surface \( S \) to be a rectangle.

However, in 2D the closed surface can be reduced to a closed line.
Surface Integrals to Line Integrals (2 of 2)

We must write our integrations as closed line integrals.

\[
\oint_S \mathbf{F} \cdot d\mathbf{r} \to \oint_C \mathbf{F} \cdot d\mathbf{l}
\]

Given that our integrating line is a rectangle, we divide this into four separate ordinary integrals.

\[
\oint_C \mathbf{F} \cdot d\mathbf{l} = \int_{x_1}^{x_2} (\cdots) \, dx + \int_{y_1}^{y_2} (\cdots) \, dy
\]

Orientation of the Surface Currents (1 of 3)

We have defined the surface currents to be flowing in a common direction.
Orientation of the Surface Currents (2 of 3)

However, this is not the inherent directionality calculated on the Yee grid.

Orientation of the Surface Currents (3 of 3)

We have to rotate the surface current vectors about the surface normal.
Rotation of Surface Current Terms

Based on this line of reasoning, we must rotate our surface current terms in two of the line segment integrals.

$$\oint \cdots d\ell' = \int_{x_1}^{x_2} (\cdots) dx + \int_{y_1}^{y_2} (\cdots) dy + \int_{x_1}^{x_2} (\cdots) dx + \int_{y_1}^{y_2} (\cdots) dy$$

Flip sign of $J_x, J_y, M_x,$ and $M_z$

Write Integrals as Standard Integrals

The last two integrals are calculated from $b$ to $a$. To arrive at a more straightforward numerical implementation, the order of integration is reversed. When this is done, the sign of the integral changes.

$$\oint \cdots d\ell' = \int_{x_1}^{x_2} (\cdots) dx + \int_{y_1}^{y_2} (\cdots) dy + \int_{x_1}^{x_2} (\cdots) dx + \int_{y_1}^{y_2} (\cdots) dy$$

Flip sign of $J_x, J_y, M_x,$ and $M_z$
Final Equations for 2D Analysis

Applying all of this to our original surface integral equations for $N(\phi)$ and $L(\phi)$, we get

$$N_x(\phi) = - \int \left[ \int_{\text{low side}} J_x e^{-\beta x} dx - \int_{\text{high side}} J_x e^{\beta x} dx \right] + \int_{\text{low side}} J_y e^{-\beta x} dy - \int_{\text{high side}} J_y e^{\beta x} dy$$

$$N_y(\phi) = \int \left[ \int_{\text{low side}} (J_x \cos \phi - J_y \sin \phi) e^{-\beta x} dx + \int_{\text{high side}} (J_x \cos \phi + J_y \sin \phi) e^{\beta x} dx \right] + \int_{\text{low side}} (J_x \cos \phi + J_y \sin \phi) e^{\beta x} dy$$

$$L_x(\phi) = \int \left[ \int_{\text{low side}} M_x e^{-\beta x} dx - \int_{\text{high side}} M_x e^{\beta x} dx \right] + \int_{\text{low side}} M_y e^{\beta x} dy - \int_{\text{high side}} M_y e^{-\beta x} dy$$

$$L_y(\phi) = \int \left[ \int_{\text{low side}} (M_x \cos \phi - M_y \sin \phi) e^{-\beta x} dx + \int_{\text{high side}} (M_x \cos \phi + M_y \sin \phi) e^{\beta x} dx \right] + \int_{\text{low side}} (M_x \cos \phi + M_y \sin \phi) e^{\beta x} dy$$

Simplifications for E and H Modes

**E-Mode**

$\mathbf{\hat{E}} = E_x \mathbf{\hat{a}}_x$

$\mathbf{\hat{H}} = H_x \mathbf{\hat{a}}_x + H_y \mathbf{\hat{a}}_y$

$E_x = E_y = H_z = 0$

$\mathbf{\hat{J}} = (n_x \mathbf{\hat{a}}_x + n_y \mathbf{\hat{a}}_y) \times (H_x \mathbf{\hat{a}}_x + H_y \mathbf{\hat{a}}_y)$

$\mathbf{\hat{M}} = - (n_x \mathbf{\hat{a}}_x + n_y \mathbf{\hat{a}}_y) \times (E_x \mathbf{\hat{a}}_x)$

$\mathbf{\hat{J}} = (n_x H_y - n_y H_x) \mathbf{\hat{a}}_z$

$\mathbf{\hat{M}} = - n_x E_z \mathbf{\hat{a}}_x + n_y E_z \mathbf{\hat{a}}_y$

$J_x = J_y = M_z = 0$

**H-Mode**

$\mathbf{\hat{E}} = E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y$

$\mathbf{\hat{H}} = H_x \mathbf{\hat{a}}_x$

$H_x = H_y = E_z = 0$

$\mathbf{\hat{J}} = (n_x \mathbf{\hat{a}}_x + n_y \mathbf{\hat{a}}_y) \times (H_x \mathbf{\hat{a}}_x)$

$\mathbf{\hat{M}} = - (n_x \mathbf{\hat{a}}_x + n_y \mathbf{\hat{a}}_y) \times (E_x \mathbf{\hat{a}}_x + E_y \mathbf{\hat{a}}_y)$

$\mathbf{\hat{J}} = n_y H_z \mathbf{\hat{a}}_x - n_x H_z \mathbf{\hat{a}}_y$

$\mathbf{\hat{M}} = (n_x E_y - n_y E_x) \mathbf{\hat{a}}_z$

$M_x = M_y = J_z = 0$
**N(θ, φ) and L(θ, φ) for the E Mode**

Given all of the simplifications on the previous slide, our expressions for \( N(θ, φ) \) and \( L(θ, φ) \) reduce to

\[
N_θ(φ) = -\int_{y \text{ low side}}^{y_1} H_x e^{-jβ'x} \cos y \, dx - \int_{y \text{ high side}}^{y_1} H_x e^{-jβ'x} \cos y \, dy + \int_{x \text{ high side}}^{x_1} H_x e^{-jβ'x} \cos y \, dx + \int_{x \text{ low side}}^{x_1} H_x e^{-jβ'x} \cos y \, dy
\]

\[
N_φ(φ) = 0
\]

\[
L_θ(φ) = \int_{y \text{ low side}}^{y_1} E_y e^{-jβ'y} \cos x \, dx + \cos φ \int_{y \text{ high side}}^{y_1} E_y e^{-jβ'y} \cos x \, dy + \int_{x \text{ low side}}^{x_1} E_y e^{-jβ'y} \cos x \, dx - \cos φ \int_{x \text{ high side}}^{x_1} E_y e^{-jβ'y} \cos x \, dy
\]

\[
L_φ(φ) = 0
\]

---

**N(θ, φ) and L(θ, φ) for the H Mode**

Given all of the simplifications on the previous slide, our expressions for \( N(θ, φ) \) and \( L(θ, φ) \) reduce to

\[
N_θ(φ) = 0
\]

\[
N_φ(φ) = \sin φ \int_{y \text{ low side}}^{y_1} H_x e^{+jβ'x} \cos y \, dx - \cos φ \int_{y \text{ high side}}^{y_1} H_x e^{+jβ'x} \cos y \, dy - \sin φ \int_{y \text{ high side}}^{y_1} H_x e^{-jβ'x} \cos y \, dx + \cos φ \int_{y \text{ low side}}^{y_1} H_x e^{-jβ'x} \cos y \, dy
\]

\[
L_θ(φ) = \int_{y \text{ low side}}^{y_1} E_y e^{+jβ'y} \cos x \, dx + \int_{y \text{ high side}}^{y_1} E_y e^{+jβ'y} \cos x \, dy + \int_{x \text{ low side}}^{x_1} E_y e^{+jβ'y} \cos x \, dx + \int_{x \text{ high side}}^{x_1} E_y e^{+jβ'y} \cos x \, dy
\]

\[
L_φ(φ) = 0
\]
**E(θ,φ) and H(θ,φ) for 2D Analysis**

Given that \( N_\phi = L_\theta = 0 \) for the E mode and \( N_\theta = L_\phi = 0 \) for the H mode, the electromagnetic far-fields for each mode are calculated as

**E Mode**

\[
E_\theta = \frac{jke^{jkr}}{4\pi r} \left( \eta N_\theta + L_\phi \right)
\]

\( E_\phi = 0 \)

\( H_\theta = 0 \)

\[
H_\phi = \frac{jke^{jkr}}{4\pi r} \left( \frac{L_\phi}{\eta} + N_\theta \right)
\]

**H Mode**

\( E_\theta = 0 \)

\[
E_\phi = \frac{jke^{jkr}}{4\pi r} \left( \eta N_\phi - L_\theta \right)
\]

\( H_\theta = \frac{jke^{jkr}}{4\pi r} \left( \frac{L_\theta}{\eta} - N_\phi \right) \)

\( H_\phi = 0 \)

---

**P_{rad}(θ,φ) and RCS(θ,φ) for 2D Analysis**

**E Mode**

\[
P_{\text{rad}}(\phi) = \frac{1}{2\eta} \left( \frac{k}{4\pi r} \right)^2 |\eta N_\phi - L_\theta|^2
\]

\[
\text{RCS}(\phi) = \frac{k^2}{8\pi\eta P_{\text{inc}}} |\eta N_\phi - L_\theta|^2
\]

**H Mode**

\[
P_{\text{rad}}(\phi) = \frac{1}{2\eta} \left( \frac{k}{4\pi r} \right)^2 |\eta N_\theta + L_\phi|^2
\]

\[
\text{RCS}(\phi) = \frac{k^2}{8\pi\eta P_{\text{inc}}} |\eta N_\theta + L_\phi|^2
\]


**Incident Power $P_{\text{inc}}$**

The power in a uniform plane wave is calculated as

$$P_{\text{inc}} = \frac{1}{2} \left|\bar{E}\right|^2 \eta = \frac{1}{2} \left|\bar{H}\right|^2 \eta$$

Assuming a unit amplitude plane wave source, $P_{\text{inc}}$ reduces to

$$P_{\text{inc}} = \begin{cases} 0.5/\eta & \text{for } \left|\bar{E}\right| = 1 \\ 0.5\eta & \text{for } \left|\bar{H}\right| = 1 \end{cases}$$

Assuming a unit amplitude plane wave source, $P_{\text{inc}}$ reduces to

$$P_{\text{inc}} = \eta_0 \left\{ \begin{array}{c} \frac{\varepsilon_r}{2} \quad \text{E Mode with } \left|\bar{E}\right|=1 \\ \frac{\mu_r}{2} \quad \text{H Mode with } \left|\bar{H}\right|=1 \end{array} \right.$$
Flow of Equations for H-Mode NF2FF

Loop over all values of $\phi$

a) Integrate to calculate $N$ and $L$ in the far field

$$N_{\phi}(\phi) = \sin \phi \int_{x_{low}}^{x_{high}} H e^{j k (x-x')} \, dx - \cos \phi \int_{y_{low}}^{y_{high}} H e^{j k (y-y')} \, dy - \sin \phi \int_{x_{low}}^{x_{high}} H e^{j k (x-x')} \, dx + \cos \phi \int_{y_{low}}^{y_{high}} H e^{j k (y-y')} \, dy$$

$$L_{\phi}(\phi) = + \int_{x_{low}}^{x_{high}} E_{x} e^{j k (x-x')} \, dx + \int_{y_{low}}^{y_{high}} E_{y} e^{j k (y-y')} \, dy - \int_{x_{low}}^{x_{high}} E_{x} e^{j k (x-x')} \, dx - \int_{y_{low}}^{y_{high}} E_{y} e^{j k (y-y')} \, dy$$

b) Calculate $E$ and $H$ in the far field from $N$ and $L$ (if desired)

$$E_{\phi} = \frac{j k e^{j \phi}}{4 \pi r} (\eta N_{\phi} - L_{\phi})$$

$$H_{\phi} = \frac{j k e^{j \phi}}{4 \pi r} \left( \frac{L_{\phi}}{\eta} - N_{\phi} \right)$$

c) Calculate $P_{rad}$ and RCS from $N$ and $L$ (if desired)

$$P_{rad}(\phi) = \frac{1}{2 \eta} \left( \frac{k}{4 \pi r} \right)^2 \left| \eta N_{\phi} + L_{\phi} \right|^2$$

$$\text{RCS}(\phi) = \frac{k^2}{8 \pi \eta P_{rad}} \left| \eta N_{\phi} + L_{\phi} \right|^2$$

Implementation
Step 1 – Setup the Grid

3D Grid  
2D Grid

- PML
- \( S \)
- TF/SF Interface
- Scattering Object

Note that the near-field surface \( S \) is located within the scattered-field region.

Step 2 – Simulate to Get Steady-State Field

3D Grid  
2D Grid

- \( S \)
- Calculate steady-state field along \( S \)

The steady-state field from FDTD is the same data that would be generated if simulated by finite-difference frequency-domain (FDFD).
Step 2 – Simulate to Get Steady-State Field

Below is MATLAB code from inside the main FDTD loop that updates the Fourier transform across the entire grid at a single frequency defined by the kernel $K$. The Fourier transform of the source at this same frequency is also calculated.

```matlab
% Update Fourier Transforms
FT_Ez = FT_Ez + (K^T*dt)*Ez;
FT_Hx = FT_Hx + (K^T*dt)*Hx;
FT_Hy = FT_Hy + (K^T*dt)*Hy;
FT_S  = FT_S  + (K^T*dt)*H0(nx2c,ny2c);
```

After the main FDTD loop has finished, the Fourier transforms are normalized to source using the follow code:

```matlab
% NORMALIZE FOURIER TRANSFORMS
FT_Ez = FT_Ez/FT_S;
FT_Hx = FT_Hx/FT_S;
FT_Hy = FT_Hy/FT_S;
```

Step 3 – Interpolate Field Quantities Along $S$

We need to interpolate the field quantities in each Yee cell at a common point. It is convenient to choose the origin of each cell as the common point.

- $E_x(i,j,k) = \frac{E_x(i-1,j,k) + E_x(i,j,k)}{2}$
- $E_y(i,j,k) = \frac{E_y(i,j-1,k) + E_y(i,j,k)}{2}$
- $E_z(i,j,k) = \frac{E_z(i,j,k-1) + E_z(i,j,k)}{2}$

- $H_x(i,j,k) = \frac{H_x(i,j-1,k-1) + H_x(i,j-1,k) + H_x(i,j,k-1) + H_x(i,j,k)}{4}$
- $H_y(i,j,k) = \frac{H_y(i-1,j,k-1) + H_y(i-1,j,k) + H_y(i,j,k-1) + H_y(i,j,k)}{4}$
- $H_z(i,j,k) = \frac{H_z(i-1,j-1,k) + H_z(i-1,j,k) + H_z(i,j-1,k) + H_z(i,j,k)}{4}$
Step 3 – Interpolate Field Quantities Along $S$

For 2D simulations, the interpolation equations reduce to

**E Mode**

$$
\vec{E}_e(i,j) = E_e(i,j) \\
\vec{H}_e(i,j) = \frac{H_e(i,j-1) + H_e(i,j)}{2} \\
\vec{H}_e(i,j) = \frac{H_e(i-1,j) + H_e(i,j)}{2}
$$

**H Mode**

\[
\vec{H}_e(i,j) = \frac{H_e(i,j-1) + H_e(i,j) + H_e(i,j-1) + H_e(i,j)}{4} \\
\vec{E}_e(i,j) = \frac{E_e(i,j-1) + E_e(i,j)}{2} \\
\vec{E}_e(i,j) = E_e(i,j)
\]

---

Step 4 – Calculate N and L by Integrating Fields Along $S$

We pick a range of values for $\phi$, usually $-\pi < \theta < \pi$. For each value of $\phi$ we integrate $E$ and $H$ around the contour $S$.

**E Mode**

\[
N_e(\phi) = -\int_{y_{\text{low side}}}^{y_{\text{high side}}} H_e e^{j\phi \cos \theta} \, dx - \int_{y_{\text{low side}}}^{y_{\text{high side}}} H_e e^{j\phi \cos \theta} \, dy + \int_{y_{\text{high side}}}^{y_{\text{low side}}} H_e e^{j\phi \cos \theta} \, dx + \int_{y_{\text{high side}}}^{y_{\text{low side}}} H_e e^{j\phi \cos \theta} \, dy \\
L_e(\phi) = -\sin \phi \int_{x_{\text{low side}}}^{x_{\text{high side}}} E_e e^{j\phi \cos \theta} \, dx + \cos \phi \int_{x_{\text{low side}}}^{x_{\text{high side}}} E_e e^{j\phi \cos \theta} \, dy + \sin \phi \int_{x_{\text{low side}}}^{x_{\text{high side}}} E_e e^{j\phi \cos \theta} \, dx - \cos \phi \int_{x_{\text{low side}}}^{x_{\text{high side}}} E_e e^{j\phi \cos \theta} \, dy
\]

**H Mode**

\[
N_h(\phi) = \sin \phi \int_{y_{\text{low side}}}^{y_{\text{high side}}} H_h e^{j\phi \cos \theta} \, dx - \cos \phi \int_{y_{\text{low side}}}^{y_{\text{high side}}} H_h e^{j\phi \cos \theta} \, dy - \sin \phi \int_{y_{\text{high side}}}^{y_{\text{low side}}} H_h e^{j\phi \cos \theta} \, dx + \cos \phi \int_{y_{\text{high side}}}^{y_{\text{low side}}} H_h e^{j\phi \cos \theta} \, dy \\
L_h(\phi) = +\int_{x_{\text{low side}}}^{x_{\text{high side}}} E_h e^{j\phi \cos \theta} \, dx + \int_{x_{\text{low side}}}^{x_{\text{high side}}} E_h e^{j\phi \cos \theta} \, dy - \int_{x_{\text{high side}}}^{x_{\text{low side}}} E_h e^{j\phi \cos \theta} \, dx - \int_{x_{\text{high side}}}^{x_{\text{low side}}} E_h e^{j\phi \cos \theta} \, dy
\]
Initialize MATLAB
- Dashboard
  - Source
  - NF2FF
  - Device
  - Grid
- Calculate Grid
  - Nx, Ny, dx, dy
- Build Device
  - ERxx, ERyy, ERzz,
  - URxx, URyy, URzz

Compute Source Parameters
- dt, tau, nxs, nys, T0, kx, ky, etc.

Compensate for Numerical Dispersion

Initialize FDTD
- Fields
- BC's
- Curls
- Integrations

This step only calculates source parameters, not the source functions. Those are calculated in the main FDTD loop.

This is the ordinary main loop for FDTD except the TF/SF and calculating Fourier transforms where the NF2FF is to be calculated.

Finish Fourier Transforms
High-Level Code for $\phi$ Loop

```matlab
% CALCULATE N AND L
r = 10*max(Sx,Sy);
PHI = linspace(-pi,pi,NPHI);
for nphi = 1 : NPHI
    % Next Phi Angle
    phi = PHI(nphi);
    rn = [ cos(phi) ; sin(phi) ];

    % ylo trapezoidal integration
    ...
    % xhi trapezoidal integration
    ...
    % yhi trapezoidal integration
    ...
    % xlo trapezoidal integration
    ...
end
```

Interpolate Fields to Origin of Yee Cells

Denormalize $E$

$$\tilde{E} = \eta_0 E$$

Initialize NF2FF

$$N_r(\theta,\phi) = N_\theta(\theta,\phi) = N_\phi(\theta,\phi) = 0$$

$$L_r(\theta,\phi) = L_\theta(\theta,\phi) = L_\phi(\theta,\phi) = 0$$

$$E_\theta(\theta,\phi) = E_\phi(\theta,\phi) = 0$$

$$H_\theta(\theta,\phi) = H_\phi(\theta,\phi) = 0$$

$$P_{rad}(\theta,\phi) = \text{RCS}(\theta,\phi) = 0$$

Integrate Around Perimeter

$$N_r, N_\theta, N_\phi, L_r, L_\theta, L_\phi$$

Calculate Fields (if desired)

$$E_\theta, E_\phi, H_\theta, H_\phi$$

Calculate RCS (if desired)

$$P_{rad}, \text{RCS}$$

Done?

No

Yes

Show Results

Finish
Example Code for Integration

\[ N_\phi(\phi) = - \int_{y_1}^{y_2} H_x e^{-j\phi \cos \theta} \, dy + \int_{y_1}^{y_2} H_y e^{-j\phi \sin \theta} \, dy + \int_{y_1}^{y_2} H_z e^{-j\phi \cos \theta} \, dy \]

\[ L_\phi(\phi) = - \sin \phi \int_{y_1}^{y_2} E_x e^{-j\phi \cos \theta} \, dy + \cos \phi \int_{y_1}^{y_2} E_y e^{-j\phi \sin \theta} \, dy - \cos \phi \int_{y_1}^{y_2} E_z e^{-j\phi \cos \theta} \, dy \]

Note: \( n_{xa} \) and \( n_{xb} \) are the left and right array indices of \( S \).
\( n_{ya} \) and \( n_{yb} \) are the top and bottom array indices of \( S \).

Simulation Examples
Example #1: Scattering from a Dielectric Cylinder

The Problem

What is the RCS of a dielectric cylinder with radius $0.5 \lambda_0$ and dielectric constant of $\varepsilon_r = 4.0$?

- $\mu_r = 1$
- $\varepsilon_r = 4.0$
- $\mu_i = 1$
- $\varepsilon_i = 1$

The Simulation (E-Mode)

The Simulation (H-Mode)
Example #1: Scattering from a Dielectric Cylinder

The Surface Currents (E-Mode)

Example #1: Scattering from a Dielectric Cylinder

The Surface Currents (H-Mode)
Example #1: Scattering from a Dielectric Cylinder

N and L Functions (E-Mode)

Example #1: Scattering from a Dielectric Cylinder

N and L Functions (H-Mode)
Example #1: Scattering from a Dielectric Cylinder

E and H Fields (E-Mode)

Example #1: Scattering from a Dielectric Cylinder

E and H Fields (H-Mode)
Example #1: Scattering from a Dielectric Cylinder

RCS (E-Mode)  
RCS (H-Mode)