Method of Lines

The method of lines (MOL) is a semi-analytical method that provides very efficient analysis of structures with long or repeated segments. It can be used to model scattering or to calculate modes in waveguides. It offers benefits when metals or high dielectric contrast is incorporated.

3D FORMULATION OF MAXWELL'S EQUATIONS

Wave Equation for i^{th} Layer

$$\frac{d^2}{d\tilde{z}^2}\begin{bmatrix}\mathbf{e}_x(\tilde{z})\\ \mathbf{e}_y(\tilde{z})\end{bmatrix}-\mathbf{\Omega}_i^2\begin{bmatrix}\mathbf{e}_x(\tilde{z})\\ \mathbf{e}_y(\tilde{z})\end{bmatrix}=\mathbf{0}$$

$$\mathbf{\Omega}_i^2=\mathbf{P}_i\mathbf{Q}_i$$

$$\mathbf{Q}_i=\begin{bmatrix}-\mathbf{D}_{x'}^H\boldsymbol{\mu}_{zz,i}^{-1}\mathbf{D}_{y'}^E & \left(\boldsymbol{\varepsilon}_{yy,i}+\mathbf{D}_{x'}^H\boldsymbol{\mu}_{zz,i}^{-1}\mathbf{D}_{x'}^E\right)\\ -\left(\boldsymbol{\varepsilon}_{xx,i}+\mathbf{D}_{y'}^H\boldsymbol{\mu}_{zz,i}^{-1}\mathbf{D}_{y'}^E\right) & \mathbf{D}_{y'}^H\boldsymbol{\mu}_{zz,i}^{-1}\mathbf{D}_{x'}^E\end{bmatrix}$$

$$\mathbf{P}_i=\begin{bmatrix}-\mathbf{D}_{x'}^E\boldsymbol{\varepsilon}_{zz,i}^{-1}\mathbf{D}_{y'}^H & \left(\boldsymbol{\mu}_{yy,i}+\mathbf{D}_{x'}^E\boldsymbol{\varepsilon}_{zz,i}^{-1}\mathbf{D}_{x'}^H\right)\\ -\left(\boldsymbol{\mu}_{xx,i}+\mathbf{D}_{y'}^E\boldsymbol{\varepsilon}_{zz,i}^{-1}\mathbf{D}_{y'}^H\right) & \mathbf{D}_{y'}^E\boldsymbol{\varepsilon}_{zz,i}^{-1}\mathbf{D}_{x'}^H\end{bmatrix}$$

Solution

$$\mathbf{\Omega}_i^2 \rightarrow \begin{matrix}\mathbf{W}_i \equiv \text{eigen-vector matrix}\\ \boldsymbol{\lambda}_i^2 \equiv \text{eigen-value matrix}\end{matrix}$$

$$\mathbf{V}=\mathbf{QW}\boldsymbol{\lambda}^{-1}$$

$$\boldsymbol{\psi}_i(\tilde{z})=\begin{bmatrix}\mathbf{e}_x(\tilde{z})\\ \mathbf{e}_y(\tilde{z})\\ \tilde{\mathbf{h}}_x(\tilde{z})\\ \tilde{\mathbf{h}}_y(\tilde{z})\end{bmatrix}=\begin{bmatrix}\mathbf{W}_i & \mathbf{W}_i\\ -\mathbf{V}_i & \mathbf{V}_i\end{bmatrix}\begin{bmatrix}e^{-\boldsymbol{\lambda}_i\tilde{z}} & \mathbf{0}\\ \mathbf{0} & e^{\boldsymbol{\lambda}_i\tilde{z}}\end{bmatrix}\begin{bmatrix}\mathbf{c}_i^+\\ \mathbf{c}_i^-\end{bmatrix}$$

2D FORMULATION OF MAXWELL'S EQUATIONS

Wave Equation for *E* Mode

$$\frac{d^2}{d\tilde{z}^2}\begin{bmatrix}\mathbf{e}_y(\tilde{z})\\ \tilde{\mathbf{h}}_x(\tilde{z})\end{bmatrix}-\mathbf{\Omega}_i^2\begin{bmatrix}\mathbf{e}_y(\tilde{z})\\ \tilde{\mathbf{h}}_x(\tilde{z})\end{bmatrix}=\mathbf{0}$$

$$\mathbf{\Omega}_i^2=\mathbf{P}_i\mathbf{Q}_i$$

$$\mathbf{P}_i=-\boldsymbol{\mu}_{xx,i}$$

$$\mathbf{Q}_i=\boldsymbol{\varepsilon}_{yy,i}+\mathbf{D}_{x'}^h\boldsymbol{\mu}_{zz,i}^{-1}\mathbf{D}_{x'}^e$$

Wave Equation for *H* Mode

$$\frac{d^2}{d\tilde{z}^2}\begin{bmatrix}\mathbf{e}_x(\tilde{z})\\ \tilde{\mathbf{h}}_y(\tilde{z})\end{bmatrix}-\mathbf{\Omega}_i^2\begin{bmatrix}\mathbf{e}_x(\tilde{z})\\ \tilde{\mathbf{h}}_y(\tilde{z})\end{bmatrix}=\mathbf{0}$$

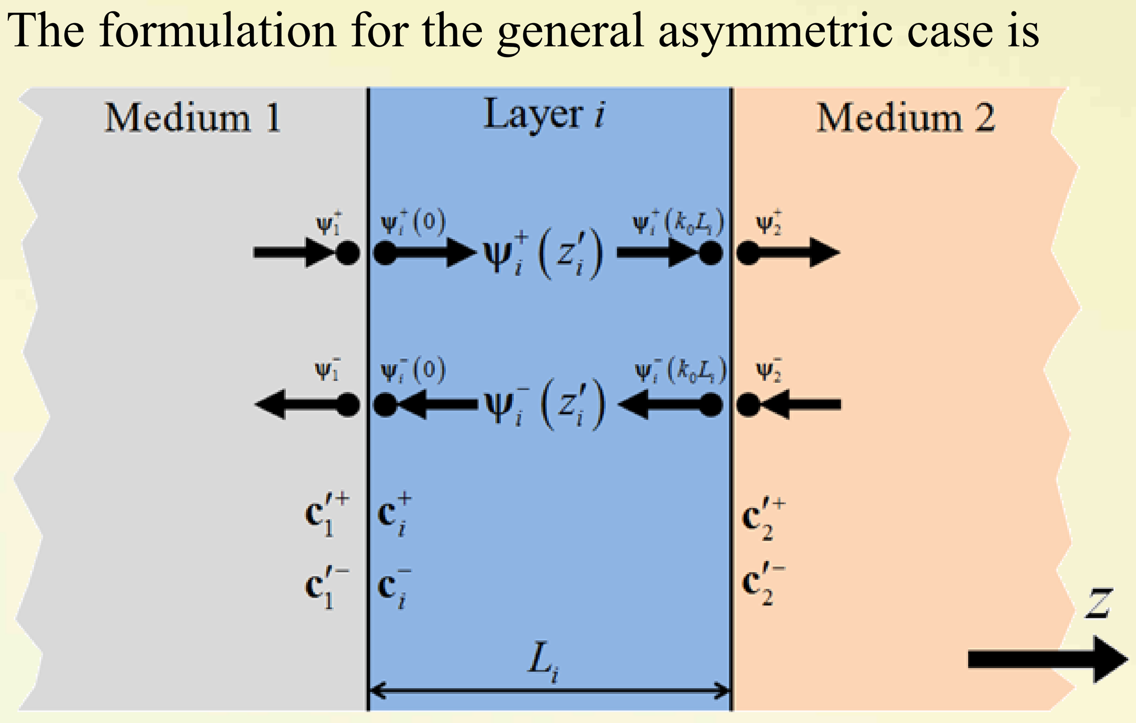
$$\mathbf{\Omega}_i^2=\mathbf{P}_i\mathbf{Q}_i$$

$$\mathbf{P}_i=-\boldsymbol{\varepsilon}_{xx,i}$$

$$\mathbf{Q}_i=\boldsymbol{\mu}_{yy,i}+\mathbf{D}_{x'}^e\boldsymbol{\varepsilon}_{zz,i}^{-1}\mathbf{D}_{x'}^h$$

THE SCATTERING MATRIX

The formulation for the general asymmetric case is



$$\begin{bmatrix}\mathbf{c}_1'^-\\ \mathbf{c}_2'^+\end{bmatrix}=\begin{bmatrix}\mathbf{S}_{11}^{(i)} & \mathbf{S}_{12}^{(i)}\\ \mathbf{S}_{21}^{(i)} & \mathbf{S}_{22}^{(i)}\end{bmatrix}\begin{bmatrix}\mathbf{c}_1'^+\\ \mathbf{c}_2'^-\end{bmatrix}$$

$$\mathbf{A}_{ij}=\mathbf{W}_i^{-1}\mathbf{W}_j+\mathbf{V}_i^{-1}\mathbf{V}_j$$

$$\mathbf{B}_{ij}=\mathbf{W}_i^{-1}\mathbf{W}_j-\mathbf{V}_i^{-1}\mathbf{V}_j$$

$$\mathbf{X}_i=e^{-\boldsymbol{\lambda}_i k_0 L_i}$$

$$\mathbf{S}_{11}^{(i)}=\left(\mathbf{A}_{i1}-\mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{B}_{i1}\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{A}_{i1}-\mathbf{B}_{i1}\right)$$

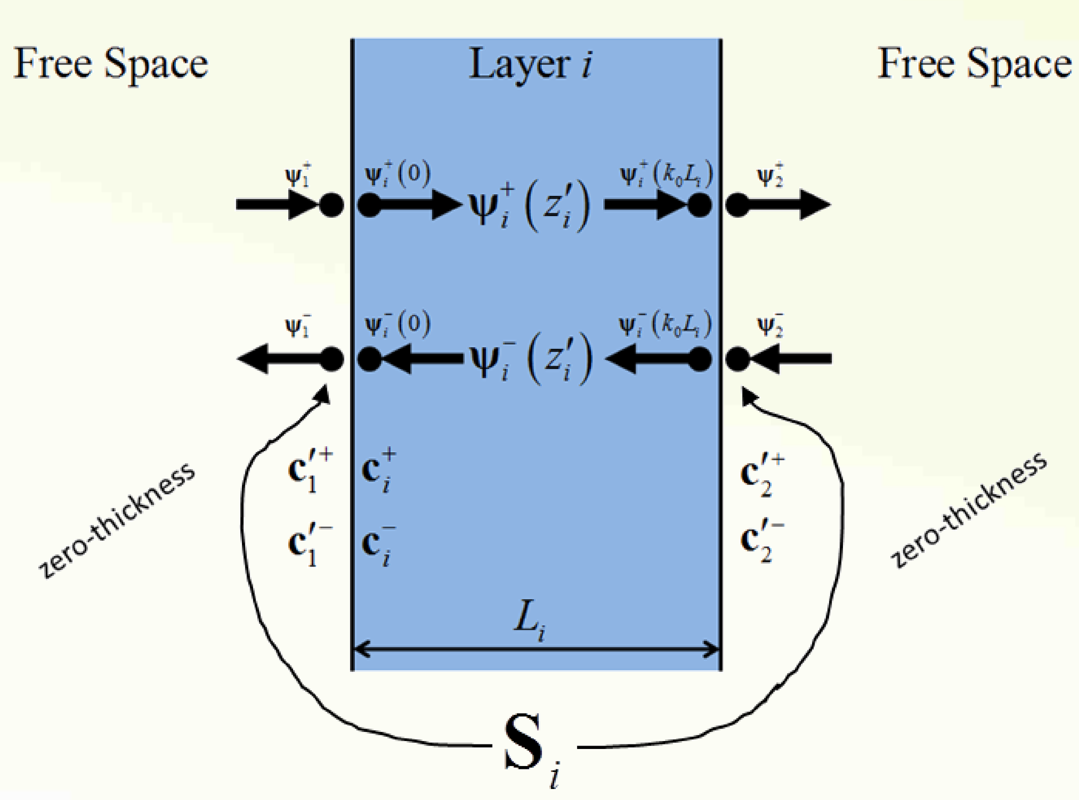
$$\mathbf{S}_{12}^{(i)}=\left(\mathbf{A}_{i1}-\mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{B}_{i1}\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_{i2}-\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{B}_{i2}\right)$$

$$\mathbf{S}_{21}^{(i)}=\left(\mathbf{A}_{i2}-\mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{B}_{i2}\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_{i1}-\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{B}_{i1}\right)$$

$$\mathbf{S}_{22}^{(i)}=\left(\mathbf{A}_{i2}-\mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{B}_{i2}\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{A}_{i2}-\mathbf{B}_{i2}\right)$$

An improved formulation offers the following benefits:

1. Symmetric scattering matrices
2. Faster the calculate.
3. More memory efficient.
4. Interchangeable scattering matrices.



$$\mathbf{A}_i=\mathbf{W}_i^{-1}\mathbf{W}_0+\mathbf{V}_i^{-1}\mathbf{V}_0$$

$$\mathbf{B}_i=\mathbf{W}_i^{-1}\mathbf{W}_0-\mathbf{V}_i^{-1}\mathbf{V}_0$$

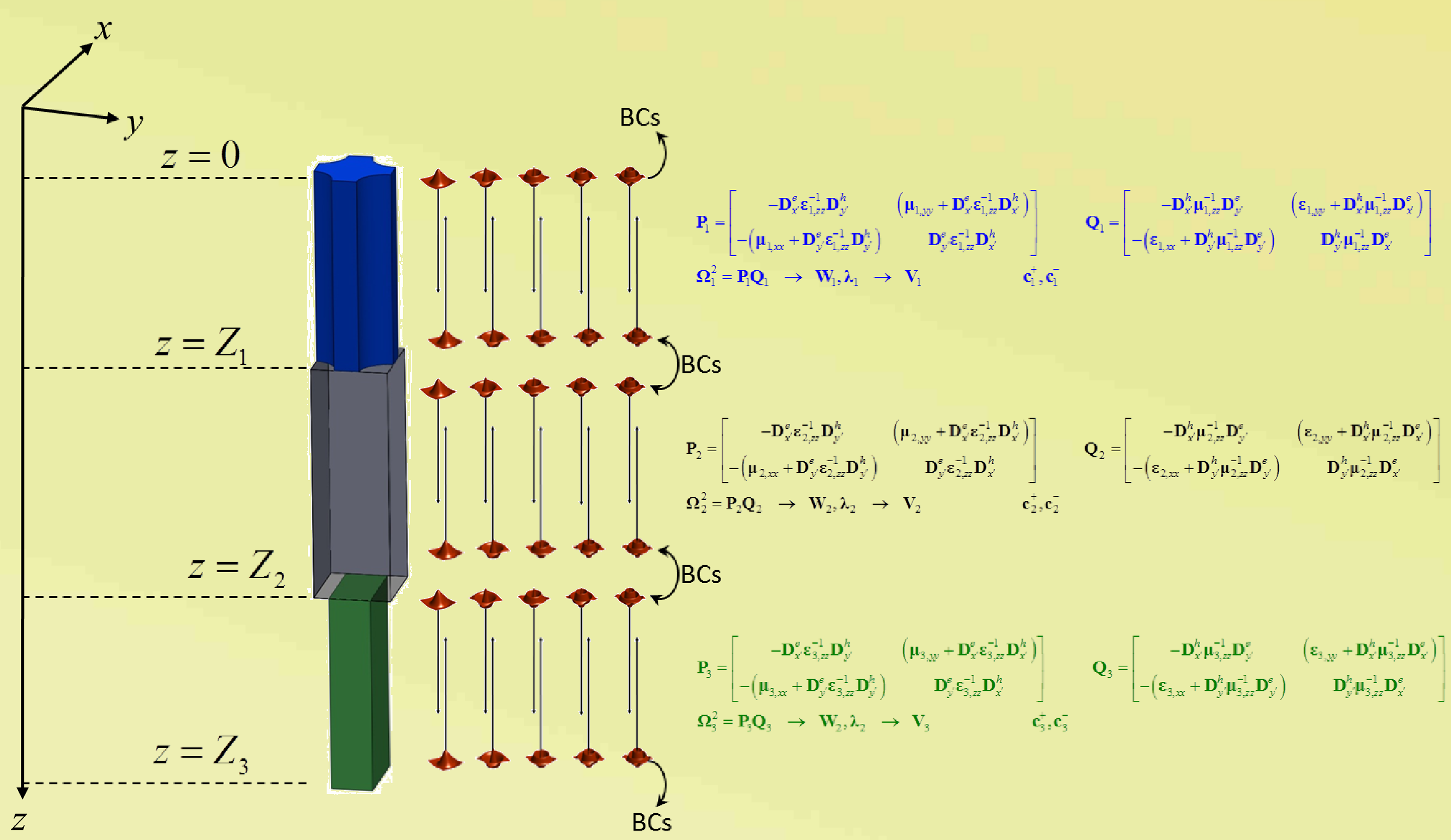
$$\mathbf{S}_{11}^{(i)}=\left(\mathbf{A}_i-\mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{A}_i-\mathbf{B}_i\right)$$

$$\mathbf{S}_{12}^{(i)}=\left(\mathbf{A}_i-\mathbf{X}_i\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{X}_i\mathbf{B}_i\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_i-\mathbf{B}_i\mathbf{A}_i^{-1}\mathbf{B}_i\right)$$

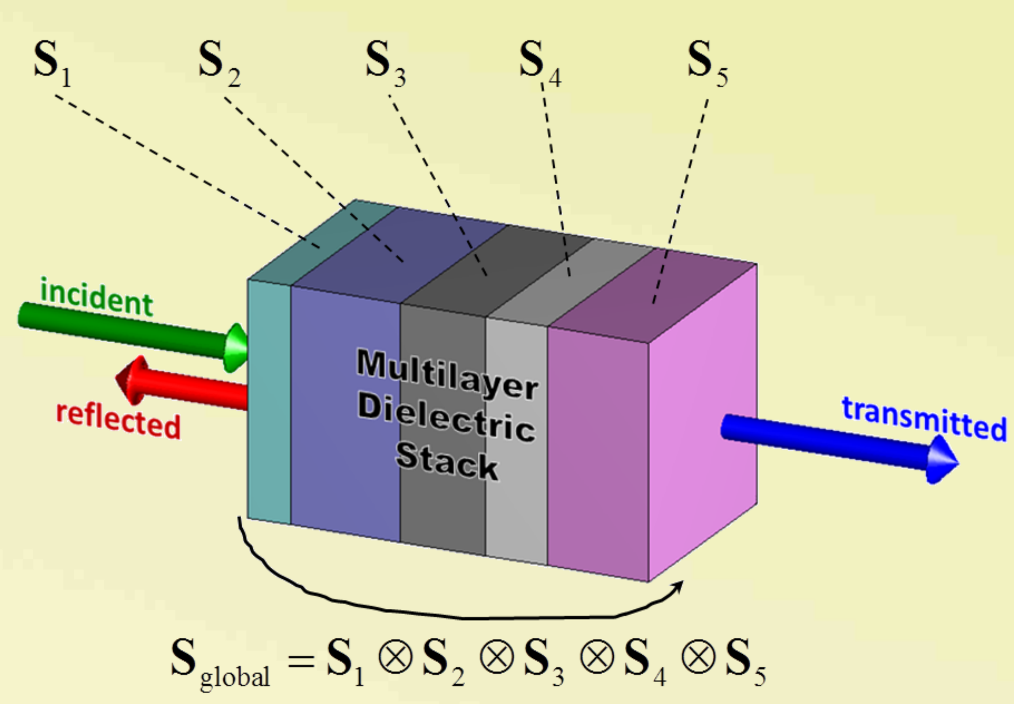
$$\mathbf{S}_{21}^{(i)}=\mathbf{S}_{12}^{(i)}$$

$$\mathbf{S}_{22}^{(i)}=\mathbf{S}_{11}^{(i)}$$

VISUALIZING THE SEMI-ANALYTICAL SOLUTION



THE SCATTERING MATRIX ALGORITHM



Redheffer Star Product:

$$\mathbf{S}=\mathbf{S}_\mathbf{a}\otimes\mathbf{S}_\mathbf{b}$$

$$\mathbf{S}_{11}=\mathbf{a}_{11}+\mathbf{a}_{12}\left(\mathbf{I}-\mathbf{b}_{11}\mathbf{a}_{22}\right)^{-1}\mathbf{b}_{11}\mathbf{a}_{21}$$

$$\mathbf{S}_{12}=\mathbf{a}_{12}\left(\mathbf{I}-\mathbf{b}_{11}\mathbf{a}_{22}\right)^{-1}\mathbf{b}_{12}$$

$$\mathbf{S}_{21}=\mathbf{b}_{21}\left(\mathbf{I}-\mathbf{a}_{22}\mathbf{b}_{11}\right)^{-1}\mathbf{a}_{21}$$

$$\mathbf{S}_{22}=\mathbf{b}_{22}+\mathbf{b}_{21}\left(\mathbf{I}-\mathbf{a}_{22}\mathbf{b}_{11}\right)^{-1}\mathbf{a}_{22}\mathbf{b}_{12}$$

CALCULATING TRANSMITTANCE AND REFLECTANCE

The source...

$$\mathbf{c}_{\text{inc}}=\mathbf{W}_{\text{ref}}^{-1}\mathbf{e}_T^{\text{inc}} \qquad \mathbf{e}_T^{\text{inc}}=\begin{bmatrix}p_x e^{-j\left(k_{x,\text{inc}}\mathbf{x}+k_{y,\text{inc}}\mathbf{y}\right)}\\ p_y e^{-j\left(k_{x,\text{inc}}\mathbf{x}+k_{y,\text{inc}}\mathbf{y}\right)}\end{bmatrix}$$

Transmitted and reflected fields...

$$\mathbf{e}_T^{\text{ref}}=\mathbf{W}_{\text{ref}}\mathbf{S}_{11}\mathbf{c}_{\text{inc}} \qquad \mathbf{e}_T^{\text{trn}}=\mathbf{W}_{\text{trn}}\mathbf{S}_{21}\mathbf{c}_{\text{inc}}$$

$$\mathbf{r}_T=\begin{bmatrix}\mathbf{r}_x\\ \mathbf{r}_y\end{bmatrix} \qquad \mathbf{t}_T=\begin{bmatrix}\mathbf{t}_x\\ \mathbf{t}_y\end{bmatrix}$$

Calculate the spatial harmonics...

$$A_{x,\text{ref}}(x,y)=r_x(x,y)\div\exp\left[-j\left(k_{x,\text{inc}}x+k_{y,\text{inc}}y\right)\right] \qquad \mathbf{r}_x=\text{FFT}_{2D}\left\{A_{x,\text{ref}}(x,y)\right\}$$

$$A_{y,\text{ref}}(x,y)=r_y(x,y)\div\exp\left[-j\left(k_{x,\text{inc}}x+k_{y,\text{inc}}y\right)\right] \qquad \mathbf{r}_x=\text{FFT}_{2D}\left\{A_{y,\text{ref}}(x,y)\right\}$$

$$A_{x,\text{trn}}(x,y)=t_x(x,y)\div\exp\left[-j\left(k_{x,\text{inc}}x+k_{y,\text{inc}}y\right)\right] \qquad \mathbf{t}_x=\text{FFT}_{2D}\left\{A_{x,\text{trn}}(x,y)\right\}$$

$$A_{y,\text{trn}}(x,y)=t_y(x,y)\div\exp\left[-j\left(k_{x,\text{inc}}x+k_{y,\text{inc}}y\right)\right] \qquad \mathbf{t}_y=\text{FFT}_{2D}\left\{A_{y,\text{trn}}(x,y)\right\}$$

Calculate the longitudinal field components...

$$\mathbf{r}_z=-\frac{\mathbf{K}_x}{\mathbf{K}_{z,\text{ref}}}\mathbf{r}_x-\frac{\mathbf{K}_y}{\mathbf{K}_{z,\text{ref}}}\mathbf{r}_y \qquad \mathbf{t}_z=-\frac{\mathbf{K}_x}{\mathbf{K}_{z,\text{trn}}}\mathbf{t}_x-\frac{\mathbf{K}_y}{\mathbf{K}_{z,\text{trn}}}\mathbf{t}_y$$

Calculate overall reflectance and transmittance...

$$\left|\vec{\mathbf{r}}\right|^2=\left|\mathbf{r}_x\right|^2+\left|\mathbf{r}_y\right|^2+\left|\mathbf{r}_z\right|^2 \qquad \mathbf{R}=\text{Re}\left[\frac{\mathbf{K}_{z,\text{ref}}}{k_{z,\text{inc}}}\right]\cdot\left|\vec{\mathbf{r}}\right|^2$$

$$\left|\vec{\mathbf{t}}\right|^2=\left|\mathbf{t}_x\right|^2+\left|\mathbf{t}_y\right|^2+\left|\mathbf{t}_z\right|^2 \qquad \mathbf{T}=\text{Re}\left[\frac{\mu_{r,\text{inc}}}{\mu_{r,\text{trn}}}\frac{\mathbf{K}_{z,\text{trn}}}{k_{z,\text{inc}}}\right]\cdot\left|\vec{\mathbf{t}}\right|^2$$

W.M. KECK CENTER
FOR 3D INNOVATION

Pioneering 21st Century Electromagnetics and Photonics

LAB