THE UNIVERSITY OF TEXAS AT EL PASO

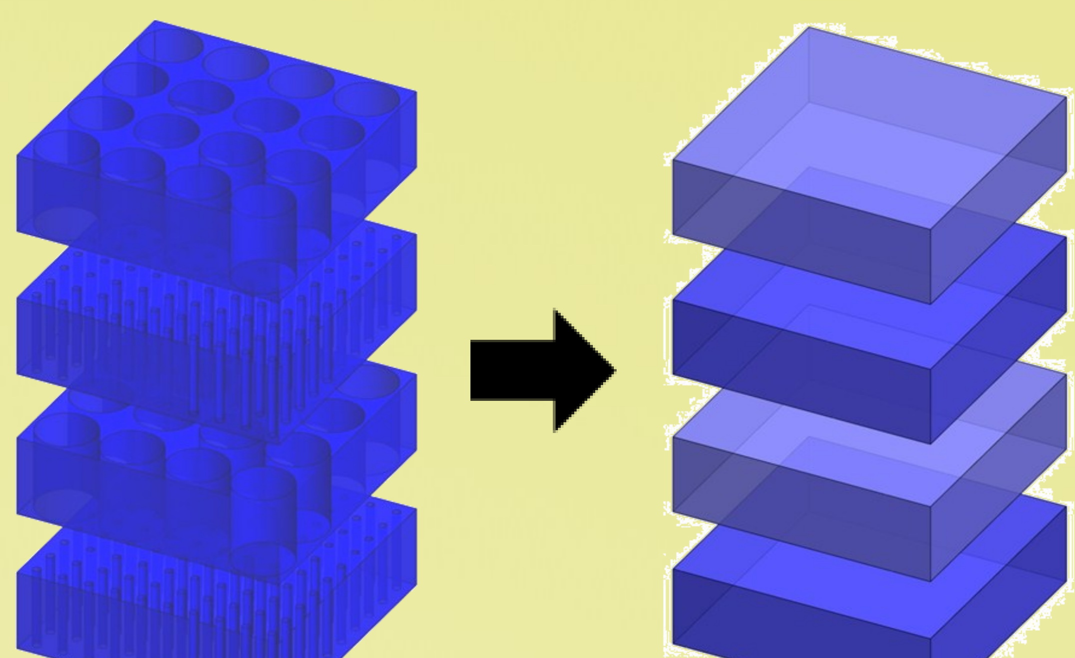
4×4 Transfer Matrix Method

Generalized to Anisotropic Layers

The transfer matrix method is an incredibly fast and efficient technique for modeling multilayer devices. Our implementation has been generalized to handle generalized anisotropic materials. It can be used to perform scattering or waveguide analysis.

3D → 1D

Using effective medium theory, it is often possible to model complex 3D devices as simple 1D structures.



MAXWELL'S EQUATIONS

$$-j\beta_y E_z - \frac{dE_y}{dz} = k_0\left(\mu_{xx}\tilde{H}_x + \mu_{xy}\tilde{H}_y + \mu_{xz}\tilde{H}_z\right) \qquad -j\beta_y \tilde{H}_z - \frac{d\tilde{H}_y}{dz} = k_0\left(\varepsilon_{xx}E_x + \varepsilon_{xy}E_y + \varepsilon_{xz}E_z\right)$$

$$\frac{dE_x}{dz} + j\beta_x E_z = k_0\left(\mu_{yx}\tilde{H}_x + \mu_{yy}\tilde{H}_y + \mu_{yz}\tilde{H}_z\right) \qquad \frac{d\tilde{H}_x}{dz} + j\beta_x \tilde{H}_z = k_0\left(\varepsilon_{yx}E_x + \varepsilon_{yy}E_y + \varepsilon_{yz}E_z\right)$$

$$-j\beta_x E_y + j\beta_y E_x = k_0\left(\mu_{zx}\tilde{H}_x + \mu_{zy}\tilde{H}_y + \mu_{zz}\tilde{H}_z\right) \qquad -j\beta_x \tilde{H}_y + j\beta_y \tilde{H}_x = k_0\left(\varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \varepsilon_{zz}E_z\right)$$

MAXWELL'S EQUATIONS IN MATRIX FORM

$$\frac{d\boldsymbol{\psi}}{d\tilde{z}} - \boldsymbol{\Omega}\boldsymbol{\psi} = 0 \qquad \boldsymbol{\psi} = \begin{bmatrix} E_x \\ E_y \\ \tilde{H}_x \\ \tilde{H}_y \end{bmatrix}$$

$$\boldsymbol{\Omega} = \begin{bmatrix}
j\left(\frac{\mu_{yz}\hat{\beta}_y}{\mu_{zz}} + \frac{\hat{\beta}_x\varepsilon_{zx}}{\varepsilon_{zz}}\right) & j\hat{\beta}_x\left(\frac{\varepsilon_{zy}}{\varepsilon_{zz}} - \frac{\mu_{yz}}{\mu_{zz}}\right) & \left(\mu_{yx} - \frac{\mu_{yz}\mu_{zx}}{\mu_{zz}} + \frac{\hat{\beta}_x\hat{\beta}_y}{\varepsilon_{zz}}\right) & \left(\mu_{yy} - \frac{\mu_{yz}\mu_{zy}}{\mu_{zz}} - \frac{\hat{\beta}_x^2}{\varepsilon_{zz}}\right) \\
j\hat{\beta}_y\left(\frac{\varepsilon_{zx}}{\varepsilon_{zz}} - \frac{\mu_{xz}}{\mu_{zz}}\right) & j\left(\frac{\mu_{xz}\hat{\beta}_x}{\mu_{zz}} + \frac{\hat{\beta}_y\varepsilon_{zy}}{\varepsilon_{zz}}\right) & \left(\frac{\mu_{xz}\mu_{zx}}{\mu_{zz}} - \mu_{xx} + \frac{\hat{\beta}_y^2}{\varepsilon_{zz}}\right) & \left(\frac{\mu_{xz}\mu_{zy}}{\mu_{zz}} - \mu_{xy} - \frac{\hat{\beta}_y\hat{\beta}_x}{\varepsilon_{zz}}\right) \\
\left(\varepsilon_{yx} - \frac{\varepsilon_{yz}\varepsilon_{zx}}{\varepsilon_{zz}} + \frac{\hat{\beta}_x\hat{\beta}_y}{\mu_{zz}}\right) & \left(\varepsilon_{yy} - \frac{\varepsilon_{yz}\varepsilon_{zy}}{\varepsilon_{zz}} - \frac{\hat{\beta}_x^2}{\mu_{zz}}\right) & j\left(\frac{\varepsilon_{yz}\hat{\beta}_y}{\varepsilon_{zz}} + \frac{\hat{\beta}_x\mu_{zx}}{\mu_{zz}}\right) & j\hat{\beta}_x\left(\frac{\mu_{zy}}{\mu_{zz}} - \frac{\varepsilon_{yz}}{\varepsilon_{zz}}\right) \\
\left(\frac{\varepsilon_{xz}\varepsilon_{zx}}{\varepsilon_{zz}} - \varepsilon_{xx} + \frac{\hat{\beta}_y^2}{\mu_{zz}}\right) & \left(\frac{\varepsilon_{xz}\varepsilon_{zy}}{\varepsilon_{zz}} - \varepsilon_{xy} - \frac{\hat{\beta}_y\hat{\beta}_x}{\mu_{zz}}\right) & j\hat{\beta}_y\left(\frac{\mu_{zx}}{\mu_{zz}} - \frac{\varepsilon_{xz}}{\varepsilon_{zz}}\right) & j\left(\frac{\varepsilon_{xz}\hat{\beta}_x}{\varepsilon_{zz}} + \frac{\hat{\beta}_y\mu_{zy}}{\mu_{zz}}\right)
\end{bmatrix}$$

NUMERICAL SOLUTION

$$\boldsymbol{\psi}(\tilde{z}) = e^{\boldsymbol{\Omega}\tilde{z}}\boldsymbol{\psi}(0) \quad \rightarrow \quad \boldsymbol{\psi}(\tilde{z}) = \mathbf{W}e^{\boldsymbol{\lambda}\tilde{z}}\mathbf{W}^{-1}\boldsymbol{\psi}(0)$$

$$\boldsymbol{\psi}(\tilde{z}) = \mathbf{W}e^{\boldsymbol{\lambda}\tilde{z}}\mathbf{c}$$

$\mathbf{W} \equiv$ eigen-vector matrix of $\boldsymbol{\Omega}$

$\boldsymbol{\lambda} \equiv$ diagonal matrix of eigen-values of $\boldsymbol{\Omega}$

SORT EIGEN-MODES

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_E^+ & \mathbf{W}_E^- \\ \mathbf{W}_H^+ & \mathbf{W}_H^- \end{bmatrix} \begin{matrix} E_x \\ E_y \\ \tilde{H}_x \\ \tilde{H}_y \end{matrix}$$

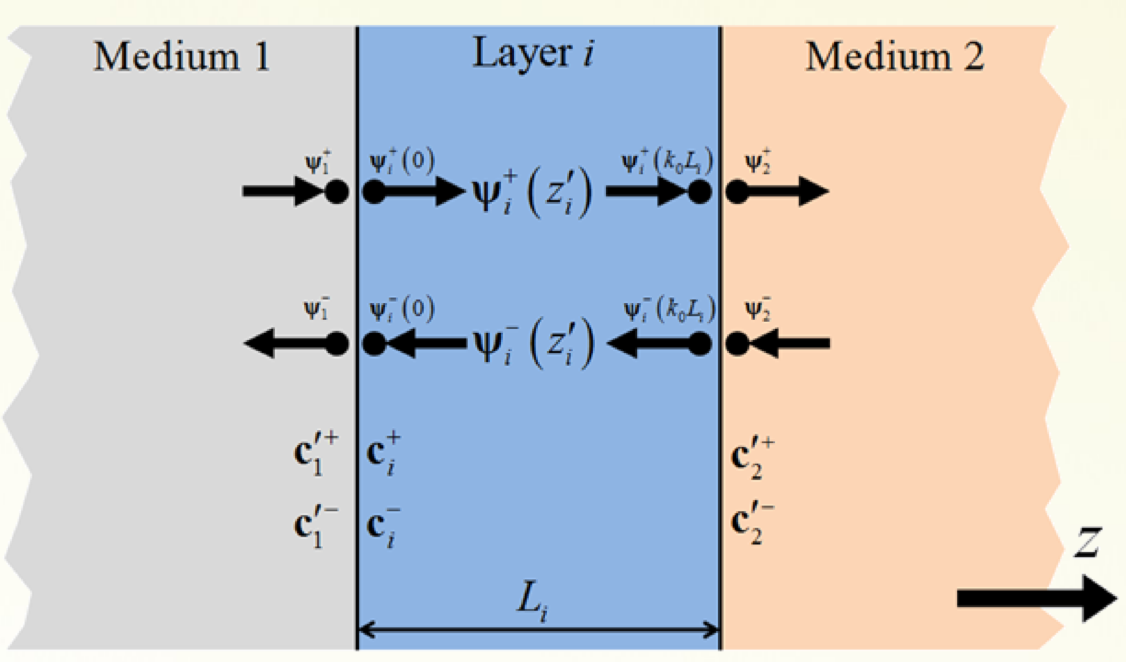
$$e^{\boldsymbol{\lambda}z'} = \begin{bmatrix} e^{\boldsymbol{\lambda}^+ z'} & \mathbf{0} \\ \mathbf{0} & e^{-\boldsymbol{\lambda}^- z'} \end{bmatrix}$$

The eigen-modes are sorted to distinguish between forward and backward propagating waves.

FIELD SOLUTION

$$\boldsymbol{\psi}(z') = \begin{bmatrix} \mathbf{W}_E^+ & \mathbf{W}_E^- \\ \mathbf{V}_H^+ & \mathbf{V}_H^- \end{bmatrix} \begin{bmatrix} e^{-\boldsymbol{\lambda}^+ z'} & \mathbf{0} \\ \mathbf{0} & e^{\boldsymbol{\lambda}^- z'} \end{bmatrix} \begin{bmatrix} \mathbf{c}^+ \\ \mathbf{c}^- \end{bmatrix}$$

THE SCATTERING MATRIX



$$\begin{bmatrix} \mathbf{c}_1'^- \\ \mathbf{c}_2'^+ \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{11}^{(i)} & \mathbf{S}_{12}^{(i)} \\ \mathbf{S}_{21}^{(i)} & \mathbf{S}_{22}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1'^+ \\ \mathbf{c}_2'^- \end{bmatrix}$$

$$\mathbf{A}_{ij} = \mathbf{W}_i^{-1}\mathbf{W}_j + \mathbf{V}_i^{-1}\mathbf{V}_j$$

$$\mathbf{B}_{ij} = \mathbf{W}_i^{-1}\mathbf{W}_j - \mathbf{V}_i^{-1}\mathbf{V}_j$$

$$\mathbf{X}_i = e^{-\boldsymbol{\lambda}_i k_0 L_i}$$

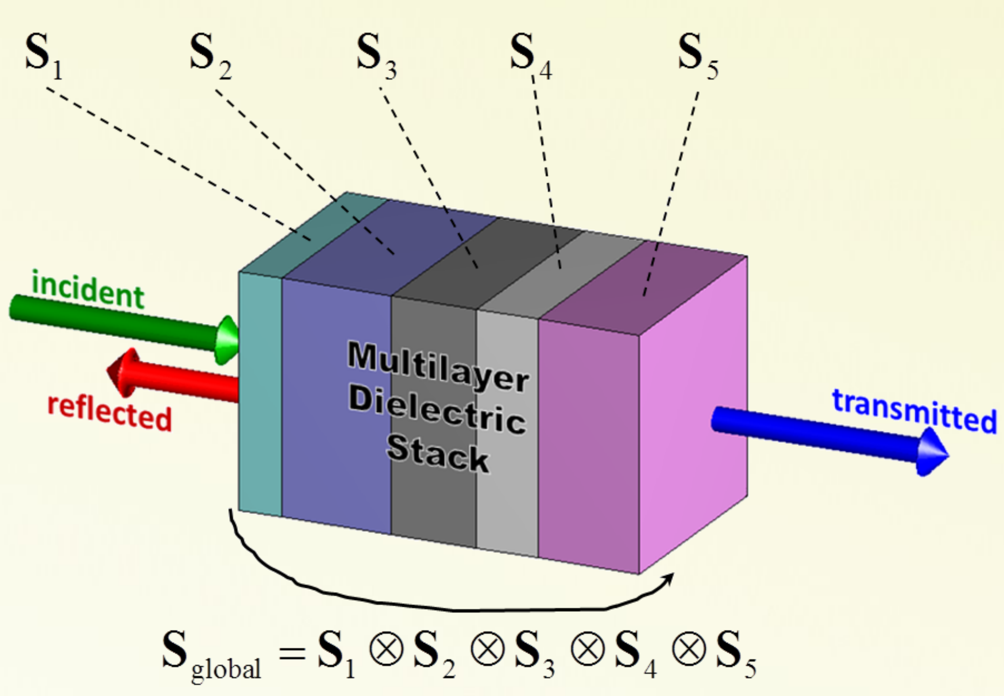
$$\mathbf{S}_{11}^{(i)} = \left(\mathbf{A}_{i1} - \mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{B}_{i1}\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{A}_{i1} - \mathbf{B}_{i1}\right)$$

$$\mathbf{S}_{12}^{(i)} = \left(\mathbf{A}_{i1} - \mathbf{X}_i\mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{X}_i\mathbf{B}_{i1}\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_{i2} - \mathbf{B}_{i2}\mathbf{A}_{i2}^{-1}\mathbf{B}_{i2}\right)$$

$$\mathbf{S}_{21}^{(i)} = \left(\mathbf{A}_{i2} - \mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{B}_{i2}\right)^{-1}\mathbf{X}_i\left(\mathbf{A}_{i1} - \mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{B}_{i1}\right)$$

$$\mathbf{S}_{22}^{(i)} = \left(\mathbf{A}_{i2} - \mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{B}_{i2}\right)^{-1}\left(\mathbf{X}_i\mathbf{B}_{i1}\mathbf{A}_{i1}^{-1}\mathbf{X}_i\mathbf{A}_{i2} - \mathbf{B}_{i2}\right)$$

THE SCATTERING MATRIX ALGORITHM



$$\mathbf{S}_{\text{global}} = \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{S}_3 \otimes \mathbf{S}_4 \otimes \mathbf{S}_5$$

Redheffer Star Product:

$$\mathbf{S} = \mathbf{S_a} \otimes \mathbf{S_b}$$

$$\mathbf{S}_{11} = \mathbf{a}_{11} + \mathbf{a}_{12}\left(\mathbf{I} - \mathbf{b}_{11}\mathbf{a}_{22}\right)^{-1}\mathbf{b}_{11}\mathbf{a}_{21}$$

$$\mathbf{S}_{12} = \mathbf{a}_{12}\left(\mathbf{I} - \mathbf{b}_{11}\mathbf{a}_{22}\right)^{-1}\mathbf{b}_{12}$$

$$\mathbf{S}_{21} = \mathbf{b}_{21}\left(\mathbf{I} - \mathbf{a}_{22}\mathbf{b}_{11}\right)^{-1}\mathbf{a}_{21}$$

$$\mathbf{S}_{22} = \mathbf{b}_{22} + \mathbf{b}_{21}\left(\mathbf{I} - \mathbf{a}_{22}\mathbf{b}_{11}\right)^{-1}\mathbf{a}_{22}\mathbf{b}_{12}$$

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