**Reading Assignment**

Read Chapters 29 and 30.

**Problem Description for Homework #9**

In this homework, you will solve the inhomogeneous Laplace’s equation to calculate the electric scalar potential that exists between two large parallel plates (i.e. a capacitor) filled with an unconventional dielectric. A diagram of the problem is provided below.

Using two voltage sources, the bottom plate is held at 1.5 V and the top plate is held at 6 V above the bottom plate. The plates are separated along the $z$-axis by a distance of 2.0 cm (i.e. $z_1 = 0$ cm and $z_2 = 2.0$ cm). The dielectric between the plates is inhomogeneous and has a relative permittivity forming a triangular profile as quantified in the figure above.

This homework will step you through the three major phases for solving this type of problem numerically. First is the *formulation* step where all the necessary equations are derived for solving the problem. Second is the *numerical implementation* step where the equations derived in the first step are solved numerically. In this homework, the finite-difference method will be implemented in MATLAB to solve the final differential equation. Last is the *visualization* step where the results of the problem are visualized in a professional manner so that meaningful conclusions can be made about the problem.
Problem #1: Formulation

For problem #1, you will derive all the necessary equations needed to implement the solution in MATLAB. All of this should be done by hand or in a word processor. MATLAB should not be used at all for Problem #1. Create a professional quality document that looks like you are explaining and formulating this problem on your own.

Do not copy/paste anything from this homework into your document. Create everything on your own.

Part 1 – Derive Differential Equation

Electrostatic problems have the following governing equations where $\vec{D}$ is the electric flux density (C/m²), $\vec{E}$ is the electric field intensity (V/m), $V$ is the electric scalar potential (volts), $\varepsilon_0$ is the permittivity of free space (F/m), $\varepsilon_r$ is the relative permittivity, $\nabla$ is the vector del operator, and $\vec{r}$ is position.

$$\nabla \cdot \vec{D}(\vec{r}) = 0 \quad \text{Gauss’ Law} \quad (1)$$

$$\vec{D}(\vec{r}) = \varepsilon_0 \varepsilon_r (\vec{r}) \vec{E}(\vec{r}) \quad \text{Constitutive Relation} \quad (2)$$

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r}) \quad \text{Electric Potential} \quad (3)$$

The differential equation for this problem is derived by first substituting Eq. (3) into Eq. (2) to eliminate $\vec{E}$, and then substituting this new expression into Eq. (1) to eliminate $\vec{D}$. The final vector differential equation is

$$\nabla \cdot \left[ \varepsilon_r (\vec{r}) \left[ \nabla V(\vec{r}) \right] \right] = 0 \quad (4)$$

Provide Eqs. (1)-(3) in your document. Then perform all of the steps to derive Eq. (4).

Part 2 – Reduce to One-Dimension

We wish to reduce this to a one-dimensional problem. This is valid as long as we are only interested in finding the electric potential far away from the edges of the capacitor. To do this mathematically, we assume the functions $\varepsilon(\vec{r})$ and $V(\vec{r})$ are only a function of $z$. Under this assumption, Eq. (4) reduces to

$$\varepsilon_r (z) \frac{d^2V(z)}{dz^2} + \frac{d\varepsilon_r (z)}{dz} \frac{dV(z)}{dz} = 0 \quad (5)$$

Add explanation and fill in the steps to derive Eq. (5) from Eq. (4).
Part 3 – Derive Matrix Equation

To solve Eq. (5) using the finite-difference method, \( \varepsilon(z) \) and \( V(z) \) are known only at discrete points along a one-dimensional grid representing the \( z \)-axis. The matrix equation derived from Eq. (5) has the following form.

\[
\begin{bmatrix} \varepsilon \end{bmatrix} \begin{bmatrix} D_z^2 \end{bmatrix} [v] + \begin{bmatrix} \varepsilon' \end{bmatrix} [D_z] [v] = [0]
\]

(6)

Fill in the steps to derive Eq. (6) from Eq. (5). You should not have to explicitly handle any finite-differences. This step should be almost effortless, but add explanations and attempt to visualize the step as best as possible.

Part 4 – \([A][x]=[0]\) Form

Equation (6) has the general form \([A][x]=[0]\), where

\[
[A][v]=[0]
\]

(7)

\[
[A]=[\varepsilon][D_z^2]+[\varepsilon'][D_z]
\]

(8)

Fill in the steps to derive this result from Eq. (6). Visualize all of the matrices and column vectors in Eq. (8). For example, \([D_z]\) might be visualized as

\[
[D_z] = \frac{1}{2 \Delta z} \begin{bmatrix}
0 & 1 \\
-1 & 0 & 1 \\
& -1 & 0 & 1 \\
& & & \ddots \\
& & & -1 & 0 & 1 \\
& & & & -1 & 0 \\
\end{bmatrix}
\]

(9)

Part 5 – Apply Boundary Conditions

Equation (7) is not yet solvable because \([v]=[A]^{-1}[0]=[0]\) is a trivial solution. It must be put into the form \([A][x]=[b]\) by applying the boundary conditions.

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
a_{21} & a_{22} & \cdots & a_{2,N-1} & a_{2,N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{N-1,1} & a_{N-1,2} & \cdots & a_{N-1,N-1} & a_{N-1,N} \\
0 & 0 & \cdots & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
V_1 \\
V_2 \\
\vdots \\
V_{N-1} \\
V_N \\
\end{bmatrix} = \begin{bmatrix}
2 \\
0 \\
\vdots \\
0 \\
6 \\
\end{bmatrix} \rightarrow V(z_1) = 1.5 \text{ V}
\]

(10)

Fill in the steps to derive this result from Eqs. (7)-(8). Describe the operations that must be performed in order to incorporate the boundary conditions. Visualize and explain this step as best as possible.
Part 6 – Solve for $[v]$

Complete your formulation by stating that the potential function $v$ is calculated from Eq. (10) as follows.

$$[v] = [A']^{-1}[b]$$

(11)

Problem #2: Implementation

In Problem #2, you will implement the finite-difference method that you formulated in Problem #1. Your implementation should have very clean MATLAB code that is well-organized and well-commented. At the end, generate a crude plot of the electric potential $V(z)$ obtained from your program. Do not worry about quality graphics yet. That will be addressed in Problem #3. Your program should follow the steps below.

Part 1 – Program Header and Dashboard

Start your MATLAB program with the following header:

```matlab
% HW9_Prob2.m
% Homework #9, Problems #2 and #3
% Computational Methods in EE
% University of Texas at El Paso
% Instructor: Dr. Raymond C. Rumpf

% INITIALIZE MATLAB
close all;
clear all;

% UNITS
centimeters = 0.01;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% DASHBOARD
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% BOUNDARY CONDITIONS
z1 = 0 * centimeters;       Va = 1.5;
z2 = 2 * centimeters;       Vb = 1.5 + 6;

% NUMBER OF GRID POINTS
Nz = 100;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% PERFORM FINITE-DIFFERENCE METHOD
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Part 2 – Calculate Grid

Calculate $\Delta z$ and calculate a one-dimensional array of points $z$ describing the position of each of the points in the interval $z_1 \leq z \leq z_2$. 
**Part 3 – Construct \( \varepsilon(z) \)**
Construct an array \( \varepsilon_R \) along \( z \) that represents the permittivity profile \( \varepsilon(z) \) shown in the figure of the capacitor.

**Part 4 – Calculate Numerical Derivative of \( \varepsilon(z) \)**
Use numerical differentiation to calculate the derivative of \( \varepsilon_R \) at EVERY point in the one-dimensional grid. Use the following finite-difference approximation to do so.

\[
\frac{df_1}{dz} \approx \frac{-25f_1 + 48f_2 - 36f_3 + 16f_4 - 3f_5}{12 \cdot \Delta z} \tag{12}
\]

\[
\frac{df_2}{dz} \approx \frac{-3f_1 - 10f_2 + 18f_3 - 6f_4 + f_5}{12 \cdot \Delta z} \tag{13}
\]

\[
\frac{df_1}{dz} \approx \frac{f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}}{12 \cdot \Delta z} \tag{14}
\]

\[
\frac{df_{N-1}}{dz} \approx \frac{-f_{N-4} + 6f_{N-3} - 18f_{N-2} + 10f_{N-1} + 3f_N}{12 \cdot \Delta z} \tag{15}
\]

\[
\frac{df_N}{dz} \approx \frac{3f_{N-4} - 16f_{N-3} + 36f_{N-2} - 48f_{N-1} + 25f_N}{12 \cdot \Delta z} \tag{16}
\]

DO NOT use derivative matrices to do this. Instead, calculate the derivatives like you did in Homework #7 for numerical differentiation.

**Part 5 – Construct Point-by-Point Multiplication Matrices**
Construct two diagonal matrices \( \varepsilon_R \) and \( \varepsilon_{RD} \) from the functions \( \varepsilon_r(z) \) and \( d\varepsilon_r(z)/dz \) respectively.

**Part 6 – Construct Derivative Operators**
Construct the two derivative matrices \( D_Z \) and \( DZ^2 \) by calling the \texttt{fdder1d()} function that you wrote in Homework #8.

**Part 7 – Build \([A]\)**
Build the matrix \( A \) you formulated in Eq. (8).

**Part 8 – Initialize \([b]\)**
Initialize a column vector \( b \) to contain all zeros.

**Part 9 – Apply Boundary Condition at \( z = z_1 \)**
Modify the matrix \( A \) and the column vector \( b \) consistent with Eq. (10) to incorporate the boundary condition at \( z = z_1 \).

**Part 10 – Apply Boundary Condition at \( z = z_2 \)**
Modify the matrix \( A \) and the column vector \( b \) consistent with Eq. (10) to incorporate the boundary condition at \( z = z_2 \).
Part 11 – Calculate the Potential $[v]$
Calculate the electric scalar potential function $v$ by solving Eq. (11).

Part 12 – Plot $V(z)$ in the Interval $z_1 \leq z \leq z_2$
Plot the scalar electric potential with $\text{plot}(z,v)$. Use the default MATLAB graphics for this. You will make it look professional in the next problem. Your plot should look like what is shown below.
Problem #3: Visualization

In this problem, you will generate professional quality graphics to visualize your problem and its solution. Your figure will contain four subplots placed horizontally next to each other: (1) the device, (2) the permittivity function \( \varepsilon(z) \), (3) the derivative of the permittivity function, and (4) the electric scalar potential \( V(z) \). In the end, your figure and results should look something like the following:

![Diagram of a capacitor with subplots]

Part 1 – Open a New Figure Window
Open a new figure window that has a white background. Size the window such that the \( y \)-axis of all the plots line up and match the device.

Part 2 – Open a Subplot to Show the Device
Select the first subplot using the \texttt{subplot()} function in MATLAB.

Part 3 – Define the RGB Values for Copper
A list of many options for RGB values for copper can be found here: http://simple.wikipedia.org/wiki/Copper_%28color%29

The values used in the solution for this homework were: \( c = [0.72 \ 0.45 \ 0.20] \);

Feel free to pick your favorite!

Part 4 – Draw the Bottom Plate
Use the \texttt{fill()} command to draw a rectangle for the bottom plate. Consider using the following limits for your rectangle. Use the copper color defined in Part 3.

\[
\begin{align*}
x_1 &= z_1 & x_2 &= z_2 \\
y_1 &= z_1 - \frac{z_2 - z_1}{20} & y_2 &= z_1
\end{align*}
\]
Part 5 – Draw the Top Plate
Use the `fill()` command to draw a rectangle for the top plate. Consider using the following limits for your rectangle. Use the copper color defined in Part 3.

\[ x_1 = z_1, \quad x_2 = z_2, \]
\[ y_1 = z_2, \quad y_2 = z_2 + \frac{z_2 - z_1}{20}. \]

Part 6 – Draw the Dielectric
Use a loop to draw the dielectric region one slice (thin rectangle) at a time along the \( z \)-axis. Use the `fill()` command to visualize each dielectric slice with a green color. Do not use an edge line for the rectangles. Scale its shade of green to convey the magnitude of the relative permittivity \( \varepsilon_r(z) \), where the darker shade corresponds to higher permittivity. Consider calculating an auxiliary function for color scale perhaps according to

\[ f(z) = 0.1 + 0.9 \frac{\varepsilon_r(z) - \min[\varepsilon_r(z)]}{\max[\varepsilon_r(z)] - \min[\varepsilon_r(z)]}, \]

then calculate the color for a slice according to

\[ c = (1-f(nz))*[1 1 1] + f(nz)*[0 0.6 0]; \]

Part 7 – Set the Graphics View for the First Subplot
Set a professional view for the device graphic by turning off the axes, giving the subplot the title “DEVICE,” and perhaps other options of your choosing.

Part 8 – Calculate the Vertical Tick Positions and Labels for the Remaining Plots
Your tick marks should go from 0 to 2.0 in steps of 0.2. Your labels should all have the same number of digits, except for zero which should only have one.

Part 9 – Plot the Relative Permittivity Function \( \varepsilon_r(z) \)
In the first subplot from the right of the device, plot the relative permittivity function \( \varepsilon_r(z) \) using a dark green line. The units for the vertical axis should be centimeters. Be sure linewidths are sufficient, but not overbearing. Label the \( x \)-axis with “\( \varepsilon_r(z) \)” and label the \( y \)-axis with “\( z \) (cm).” Set the \( y \)-axis limits so that the position of the function corresponds to the position of the devices to its left. Give the subplot the title “PERMITTIVITY.” Dress up anything else in the plot that does not look professional and ready for publication.
Part 10 – Plot the Derivative of the Relative Permittivity Function $d\varepsilon(z)/dz$

In the second subplot from the right of the device, plot the derivative of the relative permittivity function using a blue line. The units for the vertical axis should be centimeters. Be sure linewidths are sufficient, but not overbearing. Label the $x$-axis with “$d\varepsilon(z)/dz$” and label the $y$-axis with “$z$ (cm).” Set the $y$-axis limits so that the position of the function corresponds to the position of the devices to its left. Give the subplot the title “DERIVATIVE.” Dress up anything else in the plot that does not look professional and ready for publication.

Part 11 – Plot the Electric Scalar Potential $V(z)$

In the last subplot on the right, plot the electric scalar potential function $V(z)$ using a red line. The units for the vertical axis should be centimeters. Be sure linewidths are sufficient, but not overbearing. Label the $x$-axis with “$V(z)$” and label the $y$-axis with “$z$ (cm).” Set the $y$-axis limits so that the position of the function corresponds to the position of the devices to its left. Give the subplot the title “POTENTIAL.” Dress up anything else in the plot that does not look professional and ready for publication.